**First Homework, due Monday September 14, 2009**

Please solve problems 1, 4, 5, 8 on page 85-87 of L.C.Evans' book.

Solution to problem 1 on page 85 of L.C.Evans: Set $u(x,t) = e^{-ct}v(x,t)$ yields the equation

$$-ce^{-ct}v(x,t) + e^{-ct}v_t(x,t) + e^{-ct}b \cdot Dv(x,t) + ce^{-ct}v(x,t) = 0$$

with the initial condition $v(x,0) = g(x)$. Thus

$$v_t + b \cdot Dv = 0$$

which yields the solution

$$v(x,t) = g(x - bt) \ ,$$

and thus

$$u(x,t) = e^{-ct}g(x - bt) \ .$$

Solution to problem 4 on page 86 of L.C.Evans: a) As for the harmonic case consider the expression

$$\phi_x(r) = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} v(y)dS(y)$$

and differentiate with respect to $r$. You get

$$\phi'_x(r) = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} \frac{\partial v}{\partial n}(y)dS(y) = \frac{1}{|\partial B_r(x)|}\int_{B_r(x)} \Delta v(y)dS(y) \ ,$$

by Gauss' theorem. Since $\Delta v \geq 0$ this shows that $\phi'_x(r) \geq 0$ and hence

$$v(x) = \lim_{r \to 0} \phi_x(r) \leq \phi_x(r) = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} v(y)dS(y) \ .$$

Further since

$$\int_{B_r(x)} v(y)dy = \int_0^r \int_{\partial B_r(x)} v(y)dS(y)dr \geq v(x)\int_0^r |\partial B_r(x)|dr = v(x)|B_r(x)|$$

the claim follows.

b) By restricting ourselves to a connected component of $U$ we may assume that $U$ is connected. Assume that there exists a point $x_0 \in U$ where

$$v(x_0) = \max_{\overline{U}} v(x); = M \ .$$

We shall show that $v = M$ in $U$. Consider the set $C = \{x \in U : v(x) = M\}$. This set is closed, since $v$ is continuous. For $x \in C$ pick a ball $B_r(x) \subset U$ and note that

$$M = v(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} v(y) dS(y) \le M \ .$$

Hence $v(y) = M$ for all $y \in B_r(x)$ and hence $C$ is also open. Since $U$ is connected and $C$ is not empty, $C = U$. Since $v$ is continuous on $\overline{U}$, this shows that

$$\max_{\overline{U}} v(x) = \max_{\partial U} v(x) \ .$$

c) Note that

$$\nabla \phi(u) = \phi'(u) \nabla u$$

and

$$\Delta \phi(u) = \phi''(u) |\nabla u|^2 + \phi'(u) \Delta u = \phi''(u) |\nabla u|^2$$

since $u$ is harmonic. Further, since $\phi$ is convex $\Delta \phi(u) \ge 0$.

d)

$$\sum_j \partial_i (\partial_j u)^2 = 2 \sum_j \partial_i \partial_j u \partial_j u$$

$$\sum_j \partial_i^2 (\partial_j u)^2 = 2 \sum_j \partial_i^2 \partial_j u \partial_j u + 2 \sum_j (\partial_i \partial_j u)^2 \ .$$

Summing over $i$ and using that $u$ is harmonic, yields

$$\Delta |\nabla u|^2 = 2 \sum_{i,j} (\partial_i \partial_j u)^2 \ge 0 \ .$$

Solution of Problem 5 on page 86 in L.C. Evans: The solution $u = u_1 + u_2$ where $u_1$ is harmonic with boundary value $g$ and $u_2$ solves $-\Delta u_2 = f$ with boundary value 0. From the maximum/minimum principle for harmonic functions we know that

$$\max u_1 \le \max g \le \max |g| \ , \quad \min u_1 \ge \min g \ge -\max |g|$$

Hence

$$\max |u_1| \le \max |g| \ .$$

For $u_2$ we use the Green's function and write

$$u_2(x) = \int G(x,y) f(y) dS(y) \le \int G(x,y) dS(y) \max |f|$$

since the Green's function is positive. To calculate $\int G(x,y)dS(y)$ we note that this is a function $v$ that solves the equation $-\Delta v = 1$ in the unit disk and vanishes on the boundary. Hence, by uniqueness one finds

$$v(x) = \frac{1}{2n}(1 - |x|^2) \ .$$

Thus

$$\max |u_2| \leq \frac{1}{2n} \max |f| \ .$$

Solution of problem 8 on page 86 of L.C.Evans: Using Poisson's formula on the half space we can write

$$u(x) = \frac{2x_n}{|\mathcal{S}^{n-1}|} \int_{\partial \mathcal{R}^n} \frac{g(y)}{|x-y|^n} dy \ .$$

Since $g$ is continuous we know that as $x$ approaches the boundary, $u(x)$ converges to $g$ and hence $u(0) = 0$. Hence for $\lambda > 0$

$$\frac{u(\lambda e_n) - u(0)}{\lambda} = \frac{2}{|\mathcal{S}^{n-1}|} \int_{\partial \mathcal{R}^n} \frac{g(y)}{|\lambda e_n - y|^n} dy \ .$$

Next we split this integral into

$$\int_{\partial \mathcal{R}^n} \frac{g(y)}{|\lambda e_n - y|^n} dy = \int_{|y|<1} \frac{g(y)}{|\lambda e_n - y|^n} dy + \int_{|y| \geq 1} \frac{g(y)}{|\lambda e_n - y|^n} dy$$

and note that the second integral is uniformly bounded as $\lambda \to$, since $g$ is bounded. The first integral we write explicitly as

$$\int_{|y|<1} \frac{|y|}{(\lambda^2 + |y|^2)^{n/2}} dy = |\mathcal{S}^{n-2}| \int_0^1 \frac{r}{(\lambda^2 + r^2)^{n/2}} r^{n-2} dr = |\mathcal{S}^{n-2}| \int_0^1 \frac{r^{n-1}}{(\lambda^2 + r^2)^{n/2}} dr \ .$$

Since $\frac{r^{n-1}}{r^n}$ is not integrable at $r = 0$ this integral tends to infinity as $\lambda$ tends to zero.

**Problem 5:** Let $U \subset \mathcal{R}^3$ be open and $u(x) \in C^2(U)$ satisfy the equation

$$\Delta u - \mu^2 u = 0 \ ,$$

where $\mu > 0$. Show that for any ball $B_r(x) \subset U$

$$u(x) = \frac{\mu r}{\sinh(\mu r)} \frac{\int_{\partial B_r(x)} u(y) dS(y)}{|\partial B_r(x)|} \ .$$

Hint: Show that

$$\operatorname{div}\left[\frac{\sinh(\mu|x|)}{\mu|x|}\nabla u(x) - u(x)\nabla\frac{\sinh(\mu|x|)}{\mu|x|}\right] = 0$$

and integrate this identity over the ball $B_r(x)$.

Set

$$g(x) = \frac{\sinh(\mu|x|)}{\mu|x|}$$

and compute $\Delta g$. Note that $g$ is a radial function and hence, setting $r = |x|$ it suffices to calculate

$$\frac{d^2}{dr^2}g + \frac{n-1}{r}\frac{d}{dr}g = \mu^2 g \ ,$$

i.e.,

$$\Delta g = \mu^2 g \ .$$

From this we get the identity

$$\div[g\nabla u - u\nabla g] = g\Delta u - u\Delta g = 0 \ .$$

Fix $x$ and write

$$v(y) = u(x+y) \ .$$

This function satisfies the equation $\Delta v = \mu^2 v$ and claim is transformed into the statement

$$v(0) = \frac{\mu r}{\sinh(\mu r)}\int_{\partial B_r(0)} v(y)dS(y) \ .$$

To see this, integrate $0 = \div[g\nabla v - v\nabla g]$ over $B_r(0)$ and use Gauss' theorem to obtain

$$\int_{\partial B_r(0)} g(y)\frac{\partial v}{\partial n}(y)dS(y) = \int_{\partial B_r(0)} v(y)\frac{\partial g}{\partial n}(y)dS(y) \ .$$

Since $g$ is radial this reduces to

$$g(r)\int_{\partial B_r(0)}\frac{\partial v}{\partial n}(y)dS(y) = g'(r)\int_{\partial B_r(0)} v(y)dS(y) \ .$$

Divide this equation by the surface area $|\partial B_r(0)$ yields and noting, as in the second problem that

$$\frac{\int_{\partial B_r(0)}\frac{\partial v}{\partial n}(y)dS(y)}{|\partial B_r(0)|} = \frac{d}{dr}\frac{\int_{\partial B_r(0)} v(y)dS(y)}{|\partial B_r(0)|}$$

we get

$$g(r)\frac{d}{dr}\frac{\int_{\partial B_r(0)} v(y)dS(y)}{|\partial B_r(0)|} = g'(r)\frac{\int_{\partial B_r(0)} v(y)dS(y)}{|\partial B_r(0)|} \ .$$

From this one gets readily that

$$\frac{\int_{\partial B_r(0)} v(y)dS(y)}{g(r)|\partial B_r(0)|}$$

is a constant. As $r$ tends to zero this ratio converges to $v(0)$.