Second Homework, due Wednesday October 7, 2009

1) The initial temperature distribution of a rod of length is given by

$$Ax(1-x)\ ,\ 0 \le x \le 1\ ,$$

where A is a constant. Find the temperature distribution at time t when both ends of the rod are kept at zero temperature.

Solution: The Fourier Ansatz

$$\sum_{n=1}^{\infty} c_n \frac{\sin(\pi n x)}{\sqrt{2}} e^{-\pi^2 n^2 t}$$

is a solution of the heat equation with the right boundary conditions. If we note that

$$\frac{1}{2}\int_0^1 sin(\pi n x)\sin(\pi m x) dx = \delta_{m,n}$$

we have

$$c_n = \frac{A}{\sqrt{2}}\int_0^1 x(1-x)\sin(\pi n x)dx = -\frac{A}{\sqrt{2}\pi^2 n^2}\int_0^1 x(1-x)\frac{d^2}{dx^2}\sin(\pi n x)dx$$

which upon integrating by parts yields

$$\frac{\sqrt{2}A}{\pi^3 n^3}(-\cos(\pi n x)|_0^1 = \frac{\sqrt{2}A}{\pi^3 n^3}(1-(-1)^n)$$

i.e.,

$$c_{2m} = 0\ , m = 1, 2, \ldots$$

and

$$c_{2m+1} = \frac{2\sqrt{2}A}{\pi^3 n^3}\ ,\ m = 1.2.3. \ldots\ .$$

2) Let U be a bounded open and smooth domain in $\mathcal{R}^n$. Consider $u(x,t)$ the solution of the initial value problem

$$u_t = \Delta u\ \ ,\ \text{in } U \times (0,\infty)$$

$$u = f\ \ ,\ \text{on } U \times \{t = 0\}$$

$$u = 0\ \ ,\ \text{on } \partial U \times (0,\infty)\ .$$

Let $\Gamma \subset \partial U$ be a portion of the boundary. Calculate the total amount of heat that flows across Γ. Express your answer in terms of the *harmonic measure* given by

$$\Delta h = 0\ \ ,\ \text{in } U$$

$$h = 1 \quad , \text{ on } \Gamma$$

$$h = 0 \quad , \text{ on } \partial U \setminus \Gamma \ .$$

Solution: The amount of heat that flows through Γ per unit time is

$$\int_\Gamma \frac{\partial u}{\partial n} dS(y)$$

which can be written as

$$\int_{\partial U} \frac{\partial u}{\partial n} h(y) dS(y) = \int_{\partial U} \left[\frac{\partial u}{\partial n} h(y) - u(y) \frac{\partial h}{\partial n} \right] dS(y) \ ,$$

since u vanishes on ∂U. Green's second identity ten tells us that

$$\int_\Gamma \frac{\partial u}{\partial n}(y) dS(y) = \int_U \left[\Delta u(y) h(y) - u(y) \Delta h(y) \right] dS(y) = \int_U \Delta u(y) h(y) dS(y)$$

since h is harmonic in U. Using the heat equation the heat flow through Γ per unit time is

$$\int_U u_t(y) h(y) dS(y) \ .$$

We use now the (unproven) fact that $u(x,t) \to 0$ as $t \to \infty$ to integrate and obtain for the total amount of heat that flows through Γ

$$\int_0^\infty \int_U u_t(y) h(y) dS(y) = - \int_U f(y) h(y) dy \ .$$

Note that the sign is right. If f is positive then the heat flows out of U which is indicated by the minus sign.

3) Please solve Problems 11 and 16 on page 87/88 of L.C Evans' book.

Solution of problem 11 on page 87 of L.C.Evans: It is interesting to note that if $u(x,t)$ is a solution of the heat equation, then $u(\lambda^2 t, \lambda x)$ is also a solution. Here λ is any positive real number. Hence it is reasonable to look for solutions of the form $v(\frac{|x|^2}{t})$. Plugging this into the heat equation leads to

$$4zv''(z) + (2+z)v'(z) = 0$$

where $z > 0$. The converse is also obvious, namely whenever $v(z)$ satisfies the above equation, then $v(\frac{|x|^2}{t})$ satisfies the heat equation. Setting $w(z) = v'(z)$ we get

$$4zw'(z) + (2+z)w(z) = 0$$

which can be separated into

$$\frac{w'}{w} = -\frac{1}{2z} - \frac{1}{4}$$

and the solution is

$$w(z) = Ce^{-z/4}z^{-1/2}$$

which leads immediately to

$$v(z) = C\int_0^z e^{-s/4}s^{-1/2}ds + D\ .$$

Now $v(\frac{|x|^2}{t})$ is a solution of the heat equation and so is $\partial_x v(\frac{|x|^2}{t})$ which is

$$Ce^{-\frac{|x|^2}{4t}}\left(\frac{|x|^2}{t}\right)^{-1/2}\frac{2x}{t}$$

Choosing $C = \frac{1}{4\sqrt{\pi}}$ for $x > 0$ and $C = -\frac{1}{4\sqrt{\pi}}$ for $x < 0$ gets us the fundamental solution.

Solution of problem 16 on page 87 of L.C. Evans

The solution is based on the identity

$$\text{curl}(\text{curl}v) = -\Delta v + \nabla\text{div}v$$

which is easy to verify. Now $E_t = \text{curl}B$ and hence

$$\frac{\partial}{\partial t}\text{curl}E = -\Delta B + \nabla\text{div}B = -\Delta B$$

since $\text{div}B = 0$. Further since $\text{curl}E = -B_t$

$$\frac{\partial}{\partial t}\text{curl}E = -B_{tt}$$

and therefore

$$B_{tt} - \Delta B = 0\ .$$

The calculation for E is similar.

4) By descending from two to one dimension, proof d'Alembert's formula for the initial value problem

$$u_{tt} - u_{xx} = 0\ \ ,\ \text{on}\ \mathcal{R}\times(0,\infty)$$

$$u = g\ , u_t = h\ \ ,\ \text{in}\ \mathcal{R}\times\{t=0\}\ .$$

Solution: Recall that the solution for the two dimensional wave equation is given by (L.C. Evans page 74 formula (26))

$$u(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{B_t(x)} \frac{g(y)}{\sqrt{t^2 - |y-x|^2}}dy\right) + \frac{1}{2\pi}\int_{B_t(x)} \frac{h(y)}{\sqrt{t^2 - |y-x|^2}}dy$$

Now consider the solution where the initial conditions do not depend on the second variable, i.e., $g = g(x_1), h = h(x_1)$. One easily calculates

$$\int_{-t}^{t} g(y_1) \int_{-\sqrt{t^2-(x_1-y_1)^2}}^{\sqrt{t^2-(x_1-y_1)^2}} \frac{1}{\sqrt{t^2-(x_1-y_1)^2-y^2}}dydy_1 = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx \int_{x_1-t}^{x_1+t} g(z)dz$$

$$= \pi \int_{x_1-t}^{x_1+t} g(z)dz \ .$$

Hence we have the solution

$$u(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{2}\int_{x_1-t}^{x_1+t} g(z)dz\right) + \frac{1}{2}\int_{x_1-t}^{x_1+t} h(z)dz$$

$$\frac{1}{2}[g(x_1+t) + g(x_1-t)] + \frac{1}{2}\int_{x_1-t}^{x_1+t} h(z)dz$$

which is d'Alembert's formula.