Third Homework, due Wednesday November 4 , 2009

1) Solve the following initial value problems

$$a)\quad x^2u_x + y^2u_y = u^2 \ , \quad u(x,1) = \cos x \ .$$

Solution: The characteristic equations are

$$\dot{x} = x^2 \ , \quad \dot{y} = y^2 \ , \quad \dot{z} = z^2$$

with initial conditions $x(0) = x_0, y(0) = 1$ and $z(0) = \cos x_0$. The solutions are

$$y(s) = \frac{1}{1-s} \ , x(s) = \frac{x_0}{1-sx_0} \ , z(s) = \frac{\cos x_0}{1 - s\cos x_0} \ .$$

Calculating s, x_0 in terms of x, y yields

$$s = \frac{y-1}{y} \ , \quad x_0 = \frac{xy}{y-x+xy} \ .$$

Hence

$$u(x,y) = \frac{y\cos\left(\frac{xy}{y-x+xy}\right)}{y-(y-1)\cos\left(\frac{xy}{y-x+xy}\right)} \ .$$

$$b)\quad xu_x + yu_y = u \ , \quad u(x,x^2) = 1 \ , x > 0 \ .$$

Solution: The characteristic equations are

$$\dot{x} = x \ , \dot{y} = y \ , \dot{z} = z \ ,$$

with initial conditions $x(0) = x_0, y(0) = x_0^2, z(0) = 1$. The solutions are

$$x = e^s x_0 \ , \ y = e^s x_0^2 \ , \ z = e^s \ .$$

Computing s, x_0 in terms of x, y leads to the solution

$$u(x,y) = \frac{x^2}{y} \ .$$

c): Solve problem 3 c) on page 163 of L.C. Evans' book.

Solution: Characteristic equations are

$$\dot{x}_1 = x_1 \ , \ \dot{x}_2 = 2x_2 \ , \dot{x}_3 = 1 \ , \dot{z} = 3z$$

with initial conditions

$$x_1(0) = x_1^0 \ , x_2(0) = x_2^0 \ , x_3(0) = 0 \ , z(0) = g(x_1^0, x_2^0) \ .$$

The solutions:

$$x_1 = e^s x_1^0 \ , x_2 = e^{2s} x_2^0 \ , x_3 = s \ , z = e^{3s} g(x_1^0, x_2^0) \ .$$

This yields the solution

$$u(x_1, x_2, x_3) = e^{3x_3} g(x_1 e^{-x_3}, x_2 e^{-2x_3}) \ .$$

2) Do problem 6 and 7 on page 163 of L.C. Evans' book.

Solution of problem 6: Recall the Legendre transfomation of H

$$L(q) = \sup_p \{p \cdot q - H(p)\} \ .$$

Fix some p and assume that $q \in \partial H(p)$, i.e.,

$$H(r) \geq H(p) + q \cdot (r - p)$$

for all r which can be rewritten as

$$p \cdot q - H(p) \geq q \cdot r - H(r)$$

for all r. Hence we get for such q that

$$L(q) = p \cdot q - H(p) \ ,$$

or

$$L(q) + H(p) = q \cdot p \ .$$

Conversely, if this relation holds, then

$$q \cdot p - H(p) = L(q) \geq q \cdot r - H(r)$$

for all r, i.e., $q \in \partial H(p)$. The statement for L is follows since H and L play a symmetric role.

Solution of problem 7: The Hopf-Lax formula yields

$$u(x,t) = \min_y \{tL(\frac{x-y}{t}) + g(y)\} \ .$$

Assuming that L as well as g are differentiable one finds that the minimum is attained where

$$DL(\frac{x-y}{t}) = Dg(y) \ .$$

Thus, $\frac{x-y}{t}$ is in the subdifferential of $H(Dg(y))$. Hence,

$$u(x,t) = \min_{|x-y| \leq Rt} \{tL(\frac{x-y}{t}) + g(y)\} \ .$$

where

$$R = \max_y |DH(Dg(y))| \ .$$

3) Find all solutions $u(x,t)$ of the equation

$$u_t = F(u_x) \ , \quad u(x,0) = h(x) \ .$$

F and h are given differentiable functions.

Solution: The characteristic equations are

$$\dot{t} = 1 \ , \dot{x} = -F'(p_x) \ , \dot{p}_t = 0 \ , \dot{p}_x = 0 \ , \dot{z} = p_t - F'(p_x)p_x \ .$$

This problem is fully nonlinear, i.e., we don't get away by considering the projected characteristics only as we did in the other problems. The initial conditions are

$$t(0) = 0 \ , x(0) = x^0 \ , p_x(0) = h'(x^0) \ , z(0) = h(x^0) \ ,$$

and using the equation we find that

$$p_t(0) = F(h'(x^0)) \ .$$

The solutions can be expressed in terms of the variable t

$$x = x^0 - F'(h'(x_0))t \ , p_t = F(h'(x^0)) \ , p_x = h'(x^0) \ ,$$

and

$$z = h(x^0) + t(F(h'(x^0)) - F'(h'(x^0))h'(x^0)) \ .$$

The very first equation is an implicit equation for x^0 in terms of x, t. If there exists a solution x^0 in the last formula can be eliminated and yields the solution $u(x,t)$. Another way of stating this is to solve the implicit equation

$$p_x = h'(x + F'(p_x)t)$$

for p_x and then the solution is given as

$$u(x,t) = h(x + F'(p_x)t) + t(F(p_x) - F'(p_x)p_x) \ .$$

4) Solve the equation

$$\rho_t + c(1-2\rho)\rho_x = 0$$

where $\rho(x,0)$ is given by the function $f(x)$ that vanishes for negative x, equals x for x between 0 and 1, and is equals to 1 for $x > 1$. This is the traffic flow problem. Here c is the speed limit of the highway and $\rho(x,t)$ is the normalized density of cars.

Solution: The solution exists in a classical sense only up to $t = 1/(2c)$. After that shocks develop.

$$\rho(x,t) = \begin{cases} 0 & \text{if } x < ct \\ \frac{x-ct}{1-2ct} & \text{if } ct < x < 1-ct \\ 1 & \text{if } 1-ct < x \end{cases} .$$