FINAL EXAM, due December 7 , 2009

1) Consider the equation

$$u_{tt} + u_{xx} = 0 \text{ in } (0,1) \times (0,\infty) \ ,$$

$$u(0,t) = u(1,t) = 0 \ ,$$

and

$$u(x,0) = 0 \ , u_t(x,0) = \frac{1}{n}\sin(\pi n x) \ .$$

Use separation of variables to solve this initial value problem. Is this problem reasonable?

2) Find the unique entropy solution for the traffic flow problem, Problem 4 of Homework 3.

3) Let $u \in C^2(\overline{U})$ be a harmonic function with $u = f$ on ∂U. $U \subset \mathcal{R}^n$, open bounded with smooth boundary, f smooth on the boundary as well. Show that for every function $v \in C^1(\overline{U})$ with $v = f$ on ∂U

$$\int_U |Du(x)|^2 dx \leq \int_U |Dv(x)|^2 dx$$

with equality only if $v = u$. This is called Dirichlet's principle.

4) (Taken from Zachmanoglou and Thoe) Consider a random process that is independent of time and the number of events that have already taken place. Denote by $X(t)$ the total number of events during the time interval $(0,t)$ and denote the probability that $X(t) = n$ by $P_n(t)$. For any $t \geq 0$ we assume that the probability that an event occurs between time t and $t+h$ (h small) is $\lambda h + o(h)$. ($o(h)$ denotes a quantity that vanishes faster than h as $h \to 0$.) Further, we assume that the probability that more than one event happens in the time interval $(t, t+h)$ is $o(h)$. Find $P_n(t)$ for all n and t. (Note: This process is called the Poisson Process). Hint: Derive a system of differential equations for $P_n(t)$ and then derive a partial differential equation for the generating function

$$G(x,t) = \sum_{n=0}^{\infty} P_n(t) x^n \ .$$

5) (Also taken from ZT, going back to a problem of Feller). Imagine an infinite telephone network. Find the probability $P_n(t)$ that exactly n lines are in use at time $t > 0$ assuming that the initial probabilities $P_n(0)$ are given. The underlying hpotheses are

1) If a line is occupied at time t, the probability that the conversation ends during the time interval $(t, t+h)$ is $\mu h + o(h)$, μ a constant.

2) The probability of a call starting in the time interval $(t, t+h)$ is $\lambda h + o(h)$, λ a constant.

3) The probability of two or more calls starting or ending in the time interval $(t, t+h)$ is $o(h)$.

Show that $P_n(t)$ satisfy the system of ODEs

$$P_n'(t) = -(\lambda + n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t) \ .$$

Find a PDE for the generating function $G(x,t) = \sum_{n=0}^{\infty} P_n(t)x^n$. Solve this PDE and try to find $P_n(t)$.

This problem is more interesting and you may need help by consulting books, e.g., Introduction to PDE with Applications, by Zachmanoglou and Thoe, Dover. You can also come by my office for help.