Solutions for Assignment 1

1: The characterisitic equations are

$$\dot{t} = 1 \ , \ \dot{x} = z^2 \ , \ \dot{z} = 0 \ .$$

Associated first integrals are

$$z \ , \ x - z^2 t \ .$$

The solution can be found by implicitely solving the equation

$$z - g(x - z^2 t) = 0 \ .$$

For $g(x) \equiv 0$ the global solution is given by the zero function. Picking as initial conditions $g(x) = x$ yields the solution

$$u(t, x) = \frac{1}{2t}\left(\sqrt{1 + 4xt} - 1\right) \ .$$

This solution breaks down at $t = -1/4x$ for negative values of x.

A different way to understand this is to look at the characteristic equations. As an example, pick $x_0 = -5$ and hence $z_0 = -5$. Remeber that the solution of the characteristic equations once it starts on the solution surface of the PDE it always stays there. The solution of the characteristic equations are

$$x = 25t - 5 \ , \ z = -5 \ .$$

Pick now $x_0 = 0$ and hence $z_0 = 0$. The solutions are $x = 0$ and $z = 0$. Thus, when $t = 1/5$ the projections of the two curves onto the t, x plane intesect. Along the first one the solution equals -5 and along the second one the solution equals 0. Thus, the solution cannot exist globally.

2: a) The characteristic equations are

$$\dot{t} = 1 \ , \ \dot{x} = z \ , \ \dot{z} = 1 \ ,$$

and the solutions

$$t = s \ , \ z = s + z_0 \ , \ x = \frac{s^2}{2} + z_0 s + x_0 \ .$$

First integrals are

$$z - t \ , \ x - zt + \frac{t^2}{2} \ .$$

Using the initial condition we find u by solving the equation

$$z - t - (x + \frac{t^2}{2} - zt) = 0 \ ,$$

which yields

$$z = \frac{x + t + t^2/2}{1+t}|.$$

b) Solving the characteristic equations

$$\dot{t} = 1 \ , \ \dot{x} = 1 \ , \ \dot{z} = z^2 \ ,$$

leads to

$$t = s \ , \ x = s + x_0 \ , \ \frac{1}{z} + s = \frac{1}{z_0} \ .$$

Two first integrals are

$$x - t \ , \ \frac{1}{z} + t \ .$$

The solution can be found by solving the equation

$$\left(\frac{1}{z} + t\right)^{-1} - g(x-t) = 0 \ ,$$

which leads to

$$u(t,x) = \frac{g(x-t)}{1 - g(x-t)t} \ .$$

c) The char.eqns.

$$\dot{t} = 1 \ , \ \dot{x} = -xz \ , \ \dot{z} = 0$$

lead to the first integrals

$$z \ , xe^{zt} \ .$$

This leads to

$$U(t,x,z) = xe^{zt} - z$$

and the solution u is given implicitly by

$$u(t,x) = xe^{u(t,x)t} \ .$$

4: The characteristic equations are

$$\dot{t} = 1 \ , \ \dot{x} = z \ , \ \dot{z} = -x \ .$$

The solutions are

$$t = s \ ,$$

and

$$\begin{bmatrix} z(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} z_0 \\ x_0 \end{bmatrix} \ .$$

Since the initial condition is the zero function, $z_0 = 0$, and the above equation takes the form

$$z(t) = -\sin(t)x_0 \ , \ x(t) = \cos(t)x_0 \ ,$$

from which we get the solution

$$u(t, x) = -\tan(t)x \ .$$

This solution ceases to exist at time $\pi/2$.

5: The characteristic equations are

$$\dot{t} = 1 \ , \ \dot{x} = z \ , \ \dot{z} = -\sin(x) \ .$$

Recall that the solution $u(x, t)$ has the property that $u(x(s), t(s)) = z(s)$. In other words if we want to know the solution at time t and position x we find the trajectory of the characteristic equation, $x(s), t(s), z(s)$ where s is chosen such that $x(s) = x, t(s) = t$. Then the value $u(x, t)$ is $z(s)$.

Notice that the characteristic equations lead to

$$\ddot{x} = -\sin(x)$$

which is precisely the equation describing the motion of the mathematical pendulum. This equation can be solved in terms of elliptic functions but we do not have to go that far. What it important is that the solutions are periodic. The initial conditions are $t(0) = 0, z(0) = 0$ and $x(0) = x_0$ the initial point. Clearly $t = s$. The energy along any trajectory is conserved, i.e.,

$$\frac{z^2}{2} - \cos(x) = E$$

where E is a constant, the energy along the trajectory. Because of the initial conditions we are only interested in trajectories that cross the $z = 0$ axis and hence $E \in [1, -1]$. Moreover, for some such chosen E the solution is periodic say oscillating between the points $(-a, 0)$ and $(a, 0)$ where a is the positive solution of $\cos(a) = -E$. If we start at $(-a, 0)$ we reach the point $(0, \sqrt{2(E + 1)})$ after one quarter period and, starting from $(a, 0)$ we reach the point $(0, -\sqrt{2(E + 1)})$ also after a quarter period. This means that after one quarter period we have two points who have the same x-coordinate but two different z coordinates and hence our solution of the PDE ceases to exist at that point. Thus it remains to calculate the period of the trajectory for a given E. Since (choosing the half of the trajectory where $z = \dot{x} \geq 0$)

$$\dot{x} = \sqrt{2(E + \cos(x0)}$$

we get by separation of variables

$$\frac{T(E)}{2} = \int_{-a}^{a} \frac{dx}{\sqrt{2(E + \cos(x))}} = \int_{-a}^{a} \frac{dx}{\sqrt{2(\cos(x) - \cos(a))}}$$

Changing variables leads to

$$\int_{-1}^{1} \frac{adx}{\sqrt{2(\cos(xa) - \cos(a))}}$$

Using series expansions it is not very difficult to see that the period T decreases with a and converges to $T = 2\pi$ as $a \to 0$. Hence the maximal time of existence is $\pi/2$.