Solutions for Assignment 2

Problem 1

Take a point x in the interior of the quadrant $x = (x_1, x_2), x_1, x_2 > 0$. Set

$$u = (x_1, -x_2) \ , v = (-x_1, x_2) \text{ and } w = (-x_1, -x_2) \ .$$

Recall that in two dimension the fundamental solution is given by $\Phi(y - x)$, where

$$\Phi(x) = -\frac{1}{2\pi} \ln |x| \ .$$

Consider the function

$$G(y, x) = \Phi(y - x) - \Phi(y - u) - \Phi(y - v) + \Phi(y - w)$$

and note that for y on the boundary of the quadrant, i.e., $(y_1, 0)$ or $(0, y_2)$ the function $G(y, x)$ vanishes. Moreover, except for the first one all the other functions appearing in the expression for $G(y, x)$ are harmonic in the quadrant.

Problem 6 on page 87 of LCE

Using Poissons formula we get for any $r < 1$ that

$$u(x) = \frac{r^2 - |x|^2}{r|S^{n-1}|} \int_{rS^{n-1}} \frac{u(y)}{|x - y|^n} \mathrm{d}y \ ,$$

provided that $|x| < r$. By the triangle inequality we have that

$$r - |x| \leq |x - y| \leq r + |x| \ ,$$

and hence, since u is nonnegative we get that

$$\frac{r^2 - |x|^2}{r|S^{n-1}|(r + |x|)^n} \int_{rS^{n-1}} u(y)\mathrm{d}y \leq u(x) \leq \frac{r^2 - |x|^2}{r|S^{n-1}|(r - |x|)^n} \int_{rS^{n-1}} u(y)\mathrm{d}y \ .$$

Using the meanvalue property for harmonic functions we get

$$\int_{rS^{n-1}} u(y)\mathrm{d}y = r^{n-1}|S^{n-1}|u(0)$$

which yields the stated inequality.

Problem 9 on page 87 of LCE The function v is simply the odd extension of the function u defined in the upper half of the ball. The problem with directly verifying that v is harmonic is tricky since one does not know whether v is twice differentiable. What is clear, is that v is continuous on the closed unit ball since u is continuous on the closed upper half of the unit ball. Using Poisson's formula define for x inside the unit ball

$$f(x) = \frac{1-|x|^2}{|S^{n-1}|}\int_{S^{n-1}} \frac{v(y)}{|x-y|^n}\mathrm{d}y \ .$$

This function is clearly harmonic inside the unit ball and as x approaches the boundary it converges to v. Moreover, v is an odd function under reflection $y_n \to -y_n$ and if $x_n = 0$ the function $|x-y|^n$ is an even function under the reflection $y_n \to -y_n$. Thus for $x_n = 0$ the function f, being an integral of a symmetric function times an antisymmetric function, must vanish.

On the upper half ball, the function f is harmonic and since it vanishes at $x_n = 0$ it has the same boundary values as u. Hence $f = u$ on the upper half ball. On the lower half it has the same boundary value as the function

$$u_-(x) = -u(x_1, x_2, \ldots x_{n-1}, -x_n)$$

which is also harmonic. Hence $f = u_-$ on the lower half ball. Thus $f = v$ on the ball and the claim is proved.

Problem 12 on page 87 of LCE

Consider the function

$$u(x,t) = e^{-ct}v(x,t)$$

where u solves the equation of problem 12. Differentiating with respect to t yields

$$u_t = -cu + e^{-ct}v_t$$

and hence using the equation we get

$$\Delta u + f = u_t + cu = e^{-ct}v_t \ .$$

Thus, we get a new equation for v

$$v_t - \Delta v = e^{ct} f$$

and the initial condition is unchanged $v = g$ at time $t = 0$. Now, provided, f has compact support, so doeas $e^{ct}f$, and hence formula (13) on page 49 of LCE yields the solution for v, and hence for u.

Problem 13 on page 87 of LCE

As advised in the book we set $v = u - g$ and extend v to an odd function on the whole real line which we call w. This function satisfies the equation $w_t - w_{xx} = -g'$ for $x > 0$ and $w_t - w_{xx} = g'$ for $x < 0$. Hence using the formula (13) in Section 2.3 of LCE we get

$$w(x,t) = -\int_0^t \frac{1}{(4\pi(t-s))^{1/2}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-s)}}\operatorname{sign}(y)g'(s)dyds \ .$$

First we simplify

$$\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-s)}} \operatorname{sign}(y) dy$$

to

$$\int_{-x}^{\infty} e^{-\frac{y^2}{4(t-s)}} dy - \int_{x}^{\infty} e^{-\frac{y2}{4(t-s)}} dy = \int_{-x}^{x} e^{-\frac{y^2}{4(t-s)}} dy = 2(4(t-s))^{1/2} \int_0^{(4(t-s))^{-1/2}x} e^{-y^2} dy \ .$$

Hence we get

$$w(x,t) = -\frac{2}{\sqrt{\pi}} \int_0^t g'(s) \int_0^{(4(t-s))^{-1/2}x} e^{-y^2} dyds$$

which by integrating by parts turns into

$$-\frac{2}{\sqrt{\pi}} g(s) \int_0^{(4(t-s))^{-1/2}x} e^{-y^2} dy|_0^t + \frac{2}{\sqrt{\pi}} \int_0^t g(s) \frac{d}{ds} \int_0^{(4(t-s))^{-1/2}x} e^{-y^2} dyds \ .$$

The first term equals $-g(t)$, since $g(0) = 0$. The second term yields, after differentiation, the formula given in LCE on page 88.