Inhomogeneous linear PDE

Consider the PDE

$$v(x) \cdot Du(x) = f(x) \text{ in } U \subset \mathbb{R}^n \tag{1}$$

where f is a given function. As usual, we assume that $v(x)$ is a smooth vectorfield on U, so that all the theorems concerning existence, uniqueness and differentiability with respect to initial conditions of solutions hold. This equation should be solved subject to the condition that $u = g$ on some noncharacteristic hypersurface Γ.

The existence of a solution follows at once using the characteristics. Recall that the solutions of the characterisitic equations are given by a flow

$$x(t) = \Psi(t, y)$$

where y is the initial condition. Also recall that for y on a hypersurface and for s small enough the equation

$$x = \Psi(s, y)$$

can be solved for $s(x)$ and $y(x)$ and these functions are differentiable. Consider now a solution u of this initial value problem. Recall again how we solved the homogeneous equation. Fixing x close to the hypersurface Γ we run the flow backwards until the solution hits the hypersurface Γ. This determines $y(x)$ (the position where it hits the surface) and $s(x)$ (the time at which the flow reaches x starting at $y(x)$). If u solves the *homogeneous* equation

$$u(\Psi(t + s(x), y(x)))$$

is independent of t and hence

$$u(x) = u(\Psi(s(x), y(x))) = u(\Psi(-s(x) + s(x), y(x))) = u(y(x)) = g(y(x)).$$

If u solves the *inhomogeneous* equation then

$$\frac{d}{dt}u(\Psi(t+s(x), y(x))) = v(\Psi(t+s(x), y(x))) \cdot Du(\Psi(t+s(x), y(x))) = f(\Psi(t+s(x), y(x))) \ .$$

Integrating this equation form $-s(x)$ to 0 yields

$$u(x) - u(y(x)) = \int_{-s(x)}^{0} f(\Psi(t + s(x), y(x)))dt \ , \tag{2}$$

which leads immediately to the

Theorem 1
The unique solution of the initial value problem (1) is given by

$$u(x) = g(y(x)) + \int_{0}^{s(x)} f(\Psi(t, y(x)))dt \ . \tag{3}$$

This follows immediately from (2) by a change of variables and by noting that on Γ u coincides with the initial condition g.

Let us consider again the example

$$xu_y - yu_x = f(x, y)$$

with the initial condition $u(x, 0) = 0$ for $x > 0$. This problem can be solved with the following geometric steps. The solutions of the characteristic equations form concentric circles. Pick $\vec{x}$ and calculate the radius of the circle on which $\vec{x}$ lies to be $\sqrt{x^2 + y^2}$. Next one has to calculate the time it takes starting at $(\sqrt{x^2 + y^2}, 0)$ (which sits on the x- axis) to reach $\vec{x}$. Call this time $s(\vec{x})$. Now, work out formula (3). Note that this time the first integrals are not enough, we need to know the flow. Denoting the vector (x, y) by $\vec{x}$, it is elementary to see that

$$\vec{x}(t) = M(t)\vec{x}(0)$$

where

$$M(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} .$$

To find the time $s(\vec{x})$ we have to solve

$$\frac{x}{\sqrt{x^2 + y^2}} = \cos(t) \text{ and } \frac{y}{x^2 + y^2} = \sin(t) ,$$

i.e., $t = \theta(\vec{x})$, where $\theta(\vec{x})$ is the argument of the point $\vec{x}$. Now the solution is

$$u(x, y) = \int_0^{\theta(\vec{x})} f\left(\sqrt{x^2 + y^2}\cos(t), \sqrt{x^2 + y^2}\sin(t)\right) dt .$$

Finally, let us note that this problem can be solved very easily in polar coordinates.