Solutions for Test 1

Problem 1

It is easiest to do a Taylor expansion of

$$\frac{1}{1+t^2} ,$$

for small t, since we are only interested in the function for small x. The expansion for the geometric series is

$$\frac{1}{1+y} = 1 - y + y^2 - y^3 \cdots \pm y^n + R_{n+1}(c)$$

where

$$R_{n+1}(c) = (-1)^n (n-1)! \left(\frac{1}{1+c}\right)^n ,$$

Where c is some number between 0 and y. Notice that this remainder is positive if n is even and negative if n is odd. If we replace y by t^2 we get including terms up to second order in t

$$\frac{1}{1+t^2} \geq 1 - t^2 ,$$

and

$$\frac{1}{1+t^2} \leq 1 - t^2 + t^4 .$$

Integrating these inequalities yields for positive x

$$x - \frac{x^3}{3} \leq tan^{-1}(x) \leq x - \frac{x^3}{3} + \frac{x^5}{5} .$$

Thus, setting $x = 1/2$ we obtain

$$\frac{11}{24} \leq tan^{-1}(\frac{1}{2}) \leq \frac{11}{24} + \frac{1}{160} .$$

Problem 2

a) 0, b) 2, 1/24. I had previously written $-1/2$ for the a) part which is wrong. Thanks to Adam Fleming for correcting this.

Problem 3

a) The sum diverges. This can be seen as follows. For any number N we have that

$$\sum_{k=2}^{N} \frac{1}{k \ln k} \geq \int_2^N \frac{1}{x \ln x} \mathrm{d}x$$

(just draw a graph of $\frac{1}{x \ln x}$.) Substituting $u = \ln x$ in the last integral yields

$$\int_{\ln 2}^{\ln N} \frac{1}{u} \mathrm{d}u = \ln(\ln(N)) - \ln(\ln(2)) \ .$$

As N tends to infinity this term tends to infinity too.

b) The sum is absolutely convergent. This follows from the comparison test since

$$\frac{1}{1+k^2} < \frac{1}{k^2} \ ,$$

and the sum

$$\sum_{k=2}^{N} \frac{1}{k^2}$$

is convergent. (This result we did in class and follows by comparing the sum with the integral $\int 1/x^2 \mathrm{d}x$.)

c) This is a convergent geometric sum since $3/4 < 1$ and yields 3.

Problem 4

a) The integral is improper at the upper limit 1. Thus we compute for any $0 < t < 1$

$$\int_0^t \frac{x}{\sqrt{1-x^2}} \mathrm{d}x = 1 - \sqrt{1-t^2} \ .$$

As $t \to 1$ this last expression converges to 1 and hence this improper integral exists and its value is 1.

b) This problem is a bit trickier since the range of integration contains the point 1 at which the logarithm vanishes. Therefore split the integral into two pieces, one from 1/2 to 1 and the other one from 1 to 2. Lets consider the first one

$$\int_{1/2}^{1} \frac{1}{x \ln x} \mathrm{d}x \ .$$

Since this integral is improper at 1 we fix $1/2 < t < 1$ and compute

$$\int_{1/2}^{t} \frac{1}{x \ln x} \mathrm{d}x = \ln(\ln(t)) - \ln(\ln(2)) \ .$$

As t goes towards 1, $\ln(t)$ moves towards 0 and hence $\ln(\ln(t))$ moves towards $-\infty$. Similarly, the other integral does not exist either. Thus our improper integral does not exist. It contains two pieces that taken by themselves are not defined. Thus the whole integral is not defined.

c) By substitution we learn that

$$\int_0^L x e^{-x^2} \mathrm{d}x = \frac{1}{2}(1 - e^{-L^2}) \ ,$$

which tends to 1/2 as $L \to \infty$. Hence this improper integral exists.