Solutions for Test 2

Problem 1

a) Use the ratio test. The sequence converges.
b) Use the comparison test. The sum is bounded above by

$$\frac{1}{2(\ln 2)^2} + \int_2^\infty \frac{1}{x(\ln(x))^2} \mathrm{d}x \ .$$

The integral can be easily compute using the substitution $u = \ln(x)$ and equals $1/\ln(2)$.
c) Note that

$$\left(\frac{k^4}{k^4+1}\right)^{k^2} = \frac{1}{\left(1+\frac{1}{k^4}\right)^{k^2}} \ .$$

Now you know either from class or using l'Hospital's rule that

$$\lim_{k\to\infty} \left(1+\frac{1}{k^4}\right)^{k^2} = 1 \ .$$

Hence the individual terms do not go to zero and the sequence cannot converge.

Problem 2

a) The ratio test leads to computing

$$\lim_{k\to\infty} (\frac{k}{k+1})^k x = \frac{x}{e} \ .$$

Thus, the series converges absolutely in the interval $(-e, e)$.
b) Ratio test: Series converges absolutely for $|x+1| < e$ which is the same as saying that it converges on the interval $(-e-1, e-1)$.
c) Use the root test and note that

$$\lim_{k\to\infty} \left(\frac{k}{1+k}\right)^k = \frac{1}{e} \ .$$

Thus the series converges absolutely on the interval $(-e, e)$.

Problem 3

a) Note first that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^\infty x^k = \sum_{k=0}^\infty k x^{k-1} \ .$$

Therfore

$$\sum_{k=0}^\infty k\left(\frac{2}{3}\right)^k = \frac{2}{3}\sum_{k=0}^\infty k\left(\frac{2}{3}\right)^{k-1} = \frac{2}{3}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right)|_{x=2/3} \ .$$

Thus

$$\sum_{k=0}^{\infty} k \left(\frac{2}{3}\right)^k = \frac{2}{3} \frac{1}{(1-2/3)^2} = 6 \ .$$

b) The power series expansion is

$$\frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^k x^k \ .$$

c) Use that

$$e^{-x^4} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k}}{k!}$$

is an alternating series. Therfore you know that

$$e^{-x^4} \leq 1 - x^4 + \frac{1}{2} x^8 \ ,$$

and

$$e^{-x^4} \geq 1 - x^4 \ .$$

Integrating both sides of these inequalities yields

$$1 - \frac{1}{5} \leq \int_0^1 e^{-x^4} \mathrm{d}x \leq 1 - \frac{1}{5} + \frac{1}{18} \ .$$

Thus we get that the integral equals to 4/5 with an accuracy 1/18.

Problem 4

The general solution of the differential equation is

$$y(x) = \frac{1}{3} + Ce^{-3x} \ ,$$

where C is an arbitrary constant. The solution that satisfies $y(1) = 2$ is then given by

$$y(x) = \frac{1}{3} + \frac{5}{3} e^{3(1-x)} \ .$$