Solutions for Test 4

Problem 1

a) When $\vec{a}$ ad $\vec{b}$ are not proportional. b) False. c) True

d) The vectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

form a basis.

e) The first is not linear, the second is.

Problem 2

The column space is one dimensional and a basis is given by the vector

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} .$$

The nullspace is two dimensional and a basis is given by the vectors

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3/2 \\ 0 \\ -1 \end{bmatrix} .$$

Problem 3

Underdetermined system. Have to find the a vector $\vec{x}$ whose length is as small as possible and that satisfies

$$A\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} .$$

Such a vector must be perpendicular to the nullspace of A and hence in the column space of A^T. Thus

$$\vec{x} = A^T \vec{z}$$

for some vector $\vec{z}$. Therefore

$$AA^T \vec{z} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and

$$\vec{z} = \begin{bmatrix} 5/8 \\ 7/8 \end{bmatrix}$$

and hence

$$\vec{x} = \begin{bmatrix} 3/2 \\ -1/4 \\ -1/4 \end{bmatrix}$$

Problem 4

Overdetermined system. The vector $\vec{b}$ is not in the column space of the matrix A. Have to find a vector $\vec{b}^*$ in the column space of A such $\vec{b} - \vec{b}^*$ is perpendicular to the column space of A. This guarantees that the distance of the tip of $\vec{b}$ to the column space is minimal. Therefore $\vec{b} - \vec{b}^*$ is in the nullspace of the matrix A^T. Thus

$$A^T(\vec{b} - \vec{b}^*) = 0$$

and

$$\vec{b}^* = A\vec{x}^*$$

for some $\vec{x}^*$. Therefore

$$A^T\vec{b} = A^T\vec{b}^* = A^T A\vec{x}^* \ .$$

and

$$\vec{x}^* = \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix}$$

and

$$\vec{b}^* = \begin{bmatrix} 2/3 \\ 1/3 \\ -1/3 \end{bmatrix} \ .$$