

Final Exam A, Calculus II, Math1502, December 10, 2001

Name:

This test is to be taken without graphing calculators and notes of any sorts, however, a cheat sheet, standard letter format, is allowed. The allowed time is 2 hours and 50 minutes. Write answers in boxes where provided. Provide exact answers; not decimal approximations unless you are explicitly asked to do so! For example, if you mean $\sqrt{2}$ do not write 1.414.... Please show all your work because otherwise credit cannot be given.

Questions testing knowledge from Block 1

I: (15 points) Compute the integral

$$\int_0^1 \frac{\sin(y^2)}{y} dy$$

with two digits accuracy.

II: (15 points) a) Compute the limit

$$\lim_{x \rightarrow 0} \frac{\ln(1 + \sin(x^2))}{x^2} .$$

b) Find the 4-th order Taylor polynomial (around 0) of the function

$$\cos(\sin(x)) .$$

Questions testing knowledge from Block 2

III: (15 points) Find the interval of convergence, of the following power series. Include an analysis of what happens at the endpoints.

$$a) \quad \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2 + 1}} (x - 3)^k .$$

$$b) \quad \sum_{k=0}^{\infty} \frac{k^k}{(2k)^{2k}} x^k .$$

IV: (15 points) Solve the differential equations

$$xy' + 5y = \frac{1}{x} \quad \text{with initial condition} \quad y(1) = 1 ,$$

and

$$y' = \frac{y^2}{x} \quad \text{with initial condition} \quad y(1) = 1 .$$

Questions testing knowledge from Block 3

V: (15 points) Consider the following system of equations.

$$\begin{aligned}x + 2y - az &= 1 \\2x + 3y + 4z &= 2 \\3x + 7y + z &= b\end{aligned}$$

a) For which values of a and b is there a unique solution? For any of those values for a and b calculate this solution.

b) For which values of a and b , if any, does this system have no solution?

c) For which values of a and b , if any, does this system have infinitely many solutions?

VI: (5 points each, no partial credit) Find a one to one parametrization of the following systems of equations. State for each of them which variables are pivotal.

$$\begin{aligned}a) \quad & -2x + y + 2z = 2 \\& y - 2z = 2\end{aligned}$$

$$\begin{aligned}b) \quad & w - x + y - z = 1 \\& y - z = 0\end{aligned}$$

$$\begin{aligned}c) \quad & x + y - z = 1 \\& y + 2z = 0 \\& z = 1\end{aligned}$$

Questions testing knowledge from Block 4

VII: (15 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 3 & 4 & 10 \end{bmatrix}$$

a) Find $\text{Img}(A)$. Write its equation if it is a plane, give it in parametrized form if it is a line.

b) Find $\text{Ker}(A)$. Write its equation if it is a plane, give it in parametrized form if it is a line.

VIII: (20 points) Find the QR decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} .$$

Questions testing knowledge from Block 5

IX: (20 points, no partial credit) True or False: a) The matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is diagonalizable.

b) Any matrix of the form $A^T + A$ is diagonalizable.

c) Every matrix has real eigenvalues.

d) The matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is diagonalizable.

e) Every matrix has an eigenvalue.

f) Any $n \times n$ matrix A with $A^2 = I$ has 1 and -1 as the only possible eigenvalues.

X: (20 points) Compute the eigenvalues and eigenvectors of the following matrices:

a)

$$A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$$

b)

$$B = \begin{bmatrix} 8 & 3 \\ 13 & -2 \end{bmatrix}$$

c)

$$C = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

XI: (15 points) A 2×2 matrix A has the eigenvalues $\mu_1 = 1$, $\mu_2 = 4$ and the corresponding eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} .$$

a) Find U and D diagonal such that

$$A = UDU^{-1} .$$

b) Compute A .

c) Compute $\sqrt{A}$.

XII: (20 points) Find the solution of the system of differential equations

$$x' = x + 3y , \quad y' = 3x + 9y$$

with initial condition $x(0) = -2$ and $y(0) = 4$.