Solutions for Test 2

I: a) Comparison test and the bound

$$\frac{k^2}{1+k^4} \le \frac{1}{k^2}$$

yields convergence.

b) This is an alternating sum. Since

$$\frac{\ln(2k)}{(\ln(k))^2} = \frac{\ln(2)}{(\ln(k))^2} + \frac{1}{\ln(k)}$$

we see that $\frac{\ln(2k)}{(\ln(k))^2}$ tends to zero and is decreasing. Therefore the series converges.

c) Using the root test, we to compute

$$\lim_{n\to\infty} \left(\frac{n}{1+n}\right)^n = 1/e < 1$$

Therefore the series converges.

II: a) The series is a geometric series and converges precisely when

$$\frac{|x+5|}{2} < 1$$

which leads to

$$-7 < x < -3$$

for the interval of convergence.

b) Here we use the ratio test and get

$$\frac{(k+1)!|x|^{k+1}}{(2k+2)!} \frac{(2k)!}{|x|^k k!} = \frac{(k+1)|x|}{(2k+2)(2k+1)}$$

which converges to zero as $k \to \infty$ no matter how large x is. Thus, the interval of convergence is the whole real line.

c) The key observation is that

$$\sqrt{k+1} - \sqrt{k} = \frac{1}{\sqrt{k+1}+\sqrt{k}}$$

and hence using again the ratio test

$$\left(\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+2}+\sqrt{k+1}}\right)^4 |x|$$

we see that this converges to $|x|$ as $k \to \infty$. Thus we know that the series converges for $|x| < 1$. To see what happens at the endpoints note that

$$\left(\frac{1}{\sqrt{k+1}+\sqrt{k}}\right)^4 \leq \frac{1}{k^2} \ .$$

The series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges and hence the interval of convergence is given by $[-1, 1]$.

III: a) Note

$$\sum_{k=1}^{\infty} kx^k = x \sum_{k=1}^{\infty} kx^{k-1} = x \left(\frac{1}{(1-x)}\right)' = \frac{x}{(1-x)^2} \ .$$

And hence the result is

$$\frac{1}{10} \frac{1}{(1-1/10)^2} = \frac{10}{81} \ .$$

b) The power series for the exponential function is

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \ .$$

Subtracting 1 and dividing by t yields

$$\frac{e^t - 1}{t} = \sum_{k=1} \frac{t^{k-1}}{k!} \ .$$

Integrating this function from 0 to x yields

$$\sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!} \ .$$

c) We have to calculate

$$L - s_N = \sum_{k=N+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^{N+1}} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2^N} \ .$$

Thus we have to choose $N = 10$.

IV: a) The differential equation is

$$P' = 4 - \frac{P}{500} \ .$$

b) The solution is given by

$$P(t) = 2000 \left(1 - e^{-t/500}\right) \ .$$