Solutions for Test 3

I: a) The function f is linear, the function g is not.
b)

$$\begin{bmatrix} 8 \\ 7 \\ -5 \end{bmatrix}$$

c) $\sqrt{50}$ and $\sqrt{6}$.
d) The dot product vanishes, the vectors are perpendicular to each other.
e) Calculate

$$|\vec{a} + t\vec{b}|^2 = |\vec{a}|^2 + 2t\vec{a} \cdot \vec{b} + |\vec{b}|^2 \ ,$$

and note that in our example the dot product of the given vectors vanishes. Hence the minimum is attained at $t = 0$.

II: a) Recall that the matrix associated with f is given by

$$[f(\vec{e}_1), f(\vec{e}_2)].$$

Adding and subtracting $f(\vec{e}_1 + \vec{e}_2)$ and $f(\vec{e}_1 - \vec{e}_2)$ yields

$$f(2\vec{e}_1) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

and

$$f(2\vec{e}_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and hence

$$[f(\vec{e}_1), f(\vec{e}_2)] = \begin{bmatrix} -1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix} \ .$$

b) Given any vector $\vec{b}$ we can decompose it into a component perpendicular to the plane $\vec{a} \cdot \vec{x} = 1$ and one parallel to the plane. The one perpendicular is given by

$$(\vec{a} \cdot \vec{b}) \frac{\vec{a}}{|\vec{a}|^2} \tag{1}$$

and the one parallel is given by

$$\vec{b} - (\vec{a} \cdot \vec{b}) \frac{\vec{a}}{|\vec{a}|^2} \ . \tag{2}$$

Check it! The dot product of the vector (2) with $\vec{a}$ has to be zero. The length of the vector (1) is nothing bu the distance of the tip of the vector $\vec{b}$ to the plane $\vec{a} \cdot \vec{x} = 0$. But since we are interested in the distance to the plane $\vec{a} \cdot \vec{x} = 1$, we have to subtrat the distance of the plane to the origin which is given by the length of the vector

$$\frac{\vec{a}}{|\vec{a}|^2} \ . \tag{3}$$

Recall that this vector is perpendicular to the plane and its tip is on the plane. Subtracting vector (3) from vector (1) yields

$$[(\vec{a} \cdot \vec{b}) - 1]\frac{\vec{a}}{|\vec{a}|^2}$$

whose length for the vectors at hand can be calculated to be

$$\frac{3}{\sqrt{14}} \, .$$

III: a) The variables x_1 and x_2 are pivotal and x_3 is non pivotal. The one to one parametrization is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

b) All variables are pivotal. The one to one parametrization is given by

$$\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

IV: a) The system has the unique solution

$$\begin{bmatrix} 5/2 \\ 5/2 \\ 2 \end{bmatrix}$$

b) The system has infinitely many solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

c) The system has no solution.