Final Exam for Calculus II, Math 1502, December 7, 2009

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This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

Print your name (not your signature), your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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Block 1:

1: Calculate to three digits accuracy the integral

$$\int_0^1 e^{-x^2} dx \ .$$

You do not have to simplify the expression.

Solution: Since the series is alternating we have that

$$|e^{-x^2} - \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{k!}| \le \frac{x^{2(n+1)}}{(n+1)!}$$

and hence

$$|\int_0^1 e^{-x^2} dx - \sum_{k=0}^{n} (-1)^k \frac{1}{(2k+1)k!}| \le \frac{1}{(2n+3)(n+1)!}$$

If we pick $n = 5$ we find that

$$\frac{1}{13 \cdot 720} = \frac{1}{9360}$$

which might not quite do it, but $n = 6$ will be certainly sufficient. Thus

$$\sum_{k=0}^{6} (-1)^k \frac{1}{(2k+1)k!}$$

delivers the desired approximation.

2: a) Calculate the limit

$$\lim_{x \to 0} \frac{\log(1+x^2) - x^2}{x^4}$$

Solution: Either by Taylor or using l'Hôpital's rule this limit is

$$-\frac{1}{2}$$

b) Does the integral

$$\int_0^\infty \frac{\cos(\frac{1}{x})}{x^2}\,dx$$

exist? If it does, calculate it.

Solution: This integral does not exist. The problem is at 0.

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Block 2:

3: a) Find the interval of convergence, including the endpoints of the power series

$$\sum_{k=0}^{\infty} \frac{(x-2)^k}{(k+1)^2}$$

Solution: Set

$$a_k = \frac{|x-2|^k}{(k+1)^2}$$

and note that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = |x-2|$$

Hence the series converges absolutely in the interval $(1,3)$. Since

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)^2}$$

converges, the series converges absolutely on $[1,3]$.

b) Calculate the value of the sequence

$$\sum_{k=1}^{\infty} \frac{k}{2^k} \ .$$

Solution:

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1} \ ,$$

and hence

$$\sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{k}{2^{k-1}} = 2 \ .$$

4: Solve the initial value problems

$$a)\ \ y' + 2y = 4\ ,\ \ y(0) = 3\ .$$

Solution: The integrating factor is e^{2x} and hence

$$(e^{2x}y)' = 4e^{2x} = (2e^{2x})'$$

and hence

$$e^{2x}y = 2e^{2x} + c$$

so that

$$y(x) = 2 + ce^{-2x}\ .$$

The initial condition requires that $c = 1$ and hence

$$y(x) = 2 + e^{-2x}\ .$$

$$b)\ \ y' = \frac{y^2}{(1+x)^2}\ ,\ y(0) = 1\ .$$

Solution: Separation of variables lead to

$$-\left(\frac{1}{y}\right)' = -\left(\frac{1}{x}\right)'$$

and therefore

$$\frac{1}{y} = \frac{1}{1+x} + c\ .$$

The initial condition requires $c = 0$ and hence

$$y(x) = 1 + x\ .$$

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Block 3:

5: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with

$$f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } f\left(\begin{bmatrix} 7 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

Find the matrix A_f associated with f.

Solution: The matrix A_f satisfies the equation

$$A_f = \begin{bmatrix} 1 & 7 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

and hence

$$A_f = \frac{1}{10} \begin{bmatrix} -6 & 13 \\ -2 & 6 \end{bmatrix} .$$

6: Find all the solutions of the system

$$\begin{array}{cccc} x & +2y & +z & = 1 \\ 2x & & +3z & = 1 \\ -2x & +8y & -5z & = 1 \end{array} .$$

Solution: Row reducing the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & | & 1 \\ 2 & 0 & 3 & | & 1 \\ -2 & 8 & -5 & | & 1 \end{bmatrix}$$

leads to

$$\begin{bmatrix} 1 & 2 & 1 & | & 1 \\ 0 & 4 & -1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

and hence

$$\vec{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{4} \\ 1 \end{bmatrix} .$$

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Block 4:

7: Find a basis for the kernel and the image of the matrix

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 6 & 6 \\ 2 & 3 & 0 \end{bmatrix} .$$

Find also an equation describing Img(A).

Solution: Row reducing augmented matrix

$$\begin{bmatrix} 1 & 0 & -3 & | & x \\ 2 & 6 & 6 & | & y \\ 2 & 3 & 0 & | & z \end{bmatrix}$$

leads to

$$\begin{bmatrix} 1 & 0 & -3 & | & x \\ 0 & 6 & 12 & | & y-2x \\ 0 & 0 & 0 & | & -2x-y+2z \end{bmatrix}$$

Hence the equation for Img(A) is

$$-2x - y + 2z = 0 .$$

A basis is given by the vectors in the pivotal columns

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} , \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix} .$$

The vector

$$\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

spans Ker(A).

8: a) Find the QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 6 & 6 \\ 2 & 3 & 0 \end{bmatrix}$$

Solution: Gram- Schmidt leads to

$$\vec{u}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \ , \ \vec{u}_2 = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

and hence

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & 2 \\ 2 & -1 \end{bmatrix} \ .$$

$$R = \begin{bmatrix} 3 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix} \ .$$

b) Using the result of a) find the least square solution for the equation $A\vec{x} = \vec{b}$ where

$$\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ .$$

Solution: Have to solve

$$R\vec{x} = Q^T\vec{b}$$

$$Q^T\vec{b} = \frac{2}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \ .$$

The solution is

$$\vec{x} = \begin{bmatrix} -\frac{2}{9} \\ \frac{2}{9} \\ 0 \end{bmatrix} + s\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \ .$$

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Block 5:

9: True or False:

a) Every matrix is diagonalizable. FALSE

b) Every matrix whose eigenvalues have algebraic multiplicity one is diagonalizable. TRUE

c) The eigenvectors of a symmetric matrix can always be chosen to be perpendicular. TRUE

d) Two eigenvectors of a symmetric matrix may not be perpendicular? TRUE

10: Calculate the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

You do not have to calculate the eigenvectors. Is this matrix diagonalizable? Explain your answer.

Solution: The eigenvalues are $1, 2, 7, -1$. They are all different and hence the matrix is diagonalizable.

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11: Solve the differential system of differential equations

$$x' = 8x + 4y \ , \ y' = -3x + y \ ,$$

with the initial conditions $x(0) = -4, y(0) = 3$. Decide whether you want to use the 'superposition of simple solutions' method or calculating the exponential of a matrix.

Solution: Written in matrix terms we have to solve $\vec{x}' = A\vec{x}$ where

$$A = \begin{bmatrix} 8 & 4 \\ -3 & 1 \end{bmatrix}$$

The eigenvalues are $5, 4$ and the corresponding eigenvectors are

$$\begin{bmatrix} -4 \\ 3 \end{bmatrix} , \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

The initial condition immediately yields the solution

$$\vec{x}(t) = e^{5t} \begin{bmatrix} -4 \\ 3 \end{bmatrix} .$$

12: Sketch the curve given by the equation

$$4x^2 + 6xy - 4y^2 = 5 \ ,$$

by carefully calculating the eigenvalues and the eigenvectors.

Solution: Associated matrix

$$A = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$$

has the eigenvalues $5, -5$ and the corresponding eigenvectors

$$\frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix} , \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix} .$$

Curve is a hyperbola with the eigendirections as symmetry axes.