Test I for Calculus II, Math 1502, September 8, 2009

Name:

Section:

Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

Name:

Section:

Name of TA:

I: (25 points)
a) Find the 6-th order Taylor polynomial $P_6(x)$ for the function $\exp(x^2)$.

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \frac{1}{3!}\int_0^y e^z (y-z)^3 dz$$

and hence

$$P_6(x) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!}$$

b) Using the above result, compute an approximate value for

$$\int_0^1 \frac{\exp(x^2) - 1 - x^2}{x^4} dx \ .$$

Approximate value is given by

$$\int_0^1 \frac{\frac{x^4}{2} + \frac{x^6}{3!}}{x^4} dx = \int_0^1 \left(\frac{1}{2} + \frac{x^2}{3!}\right) dx = \frac{1}{2} + \frac{1}{3 \times 3!} = \frac{5}{9}$$

c) Give an estimate on how accurate the value computed in b) approximates the integral.

$$\frac{1}{3!}\int_0^{x^2} e^z (x^2 - z)^3 dz \le \frac{e}{3!}\int_0^{x^2} (x^2 - z)^3 dz$$

since $x^2 \le 1$ (we integrate up to 1!). The last expression equals

$$\frac{ex^8}{4!} \ .$$

Hence the error is bounded by

$$\int_0^1 \frac{ex^8}{x^4 4!} dx = \int_0^1 \frac{ex^4}{4!} dx = \frac{e}{5!}$$

Finally, since $e < 3$ we have

$$0 \le \int_0^1 \frac{\exp(x^2) - 1 - x^2}{x^4} dx - \frac{5}{9} \le \frac{1}{40} \ .$$

Name:

Section:

Name of TA:

II: (25 points) Compute the limits:
a)

$$\lim_{x\to 0}\left(\frac{\sin(x)}{2x}\right)^2$$

Answer:

$$\frac{1}{4}$$

since

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1$$

b)

$$\lim_{n\to\infty}(1+e^{-n})^{e^n}$$

Answer:

$$e$$

since as $n\to\infty$ we have that $x=e^n$ also tends to ∞. Hence the limit is the same as

$$\lim_{x\to\infty}(1+\frac{1}{x})^x$$

which is e.
c)

$$\lim_{x\to 0}\frac{\int_0^x[e^{-y^2}-1+y^2]dy}{x^5}$$

The answer is

$$\frac{1}{5}$$

either by expanding the function e^{-y^2} according to Taylor or by repeated use of l'Hôpital's rule.

Name:

Section:

Name of TA:

III: (25 points) Decide which of the following improper integrals exists and compute its values if it exists:
a)

$$\int_0^\infty \frac{1}{\sqrt{1+x^2}} \mathrm{d}x$$

Does not exist. The function

$$\frac{1}{\sqrt{1+x^2}}$$

behaves as $1/x$ for large x and this function is not integrable at infinity. A rigorous way of seeing this is to compute

$$\frac{1}{\sqrt{1+x^2}} - \frac{1}{x} = \frac{x - \sqrt{1+x^2}}{x\sqrt{1+x^2}} = \frac{-1}{x\sqrt{1+x^2}(x+\sqrt{1+x^2}}$$

Hence

$$\int_1^L \frac{1}{\sqrt{1+x^2}} dx = \int_1^L \frac{1}{x} dx - \int_1^L \frac{1}{x\sqrt{1+x^2}(x+\sqrt{1+x^2}} dx$$

The last integrand is bounded by $1/x^3$ which is integrable.
b)

$$\int_0^1 \frac{e^x}{\sqrt{e^x - 1}} \mathrm{d}x$$

This integral exists. For small x the integrand behaves as

$$\frac{e^x}{\sqrt{1+x-1}} = \frac{e^x}{\sqrt{x}}$$

which is integrable. Another, more rigorous, way is to calculate with $u = e^x$,

$$\int_\varepsilon^1 \frac{e^x}{\sqrt{e^x - 1}} \mathrm{d}x = \int_{e^\varepsilon}^e \frac{1}{\sqrt{u-1}} \mathrm{d}u = 2\left(\sqrt{e-1} - \sqrt{e^\varepsilon - 1}\right)$$

As ε converges to zero this expression has the limit

$$2\sqrt{e-1}\ .$$

c)

$$\int_0^\infty \frac{1}{x+x^{-1}}\mathrm{d}x$$

Does not exist. We have to compute the integral

$$\int_0^L \frac{1}{x+x^{-1}}\mathrm{d}x = \int_0^L \frac{x}{x^2+1}\mathrm{d}x = \frac{1}{2}\log(1+x^2)\Big|_0^L = \frac{1}{2}\log(1+L^2)$$

As $L \to \infty$ this tends to infinity too.

Name:

Section:

Name of TA:

IV: (25 points) Which of the following series is convergent or divergent.
a)

$$\sum_{k=1}^{\infty} k e^{-k^2} \ .$$

Convergent, since the integral

$$\int_1^{\infty} x e^{-x^2} \, dx = \frac{1}{2e}$$

exists.
b)

$$\sum_{k=2}^{\infty} \frac{k}{k^2 - 1} \ .$$

Does not exist, since the function

$$\frac{x}{x^2 - 1}$$

is not integrable on $[2, \infty)$. c) Does the following series converge and if it does, evaluate it.

$$\sum_{k=2}^{\infty} 2^k e^{-k}$$

Since $e > 2.7 > 2$ we have that

$$\frac{2}{e} < 1 \ .$$

Hence the series is convergent and we have that

$$\sum_{k=2}^{\infty} 2^k e^{-k} = \sum_{k=2}^{\infty} \left(\frac{2}{e}\right)^k = \left(\frac{2}{e}\right)^2 \sum_{k=0}^{\infty} \left(\frac{2}{e}\right)^k = \left(\frac{2}{e}\right)^2 \frac{1}{1 - \frac{2}{e}}$$