Test II for Calculus II, Math 1502, September 22, 2009

Print Name:

Section:

Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

Print your name (not your signature), your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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I: a) Let $\sum_{k=1}^{\infty} a_k x^k$ be a power series of which you only know that it converges at $x = c > 0$. Does it necessarily follow that

i) (3 points) The series $\sum_{k=1}^{\infty} k^2 a_k x^k$ converges for $x = c$? NO

ii) (3 points) The series $\sum_{k=1}^{\infty} k^4 a_k x^k$ converges for $|x| < c$? YES

iii) (3 points) The series $\sum_{k=1}^{\infty} \frac{a_k}{k^2} x^k$ converges absolutely at c? YES

iv) (3 points) The series $\sum_{k=1}^{\infty} \frac{a_k}{k^2} x^k$ diverges for $|x| > c$? NO

You do not have to explain your answer and there is no partial credit.

b) (13 points) Estimate the difference between the limit of the series

$$\sum_{k=0}^{\infty} (-1)^k \frac{k!}{(3k)!}$$

and s_3, the sum up to $k = 3$. You do not have to simplify your answer.

The series is an alternating series and converges to some real number L. Now by the estimates about alternating series

$$|L - s_3| < \frac{4!}{(12)!}$$

So

$$\frac{4!}{(12)!}$$

estimates the difference between L and s_3.

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II: (15 points) Find the first three digits after the decimal point of the integral

$$\int_0^1 J_0(x)dx \ ,$$

where $J_0(x) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{2^k(k!)^2} x^k$. ($J_0(x)$ is called the Bessel function of order 0.) You do not have to simplify your answer.

$$\int_0^1 J_0(x)dx = \sum_{k=0}^{\infty}(-1)^k \frac{1}{2^k(k!)^2} \int_0^1 x^k dx$$

$$= \sum_{k=0}^{\infty}(-1)^k \frac{1}{(k+1)2^k(k!)^2}$$

This an alternating series and if we denote its limit by L we have that

$$|L - s_n| \le \frac{1}{(n+2)2^{n+1}((n+1)!)^2}$$

Choosing $n = 2$ yields the estimate

$$|L - s_2| \le \frac{1}{32 \times 36}$$

which does not quite do it. Choosing $n = 3$ we get

$$|L - s_3| \le \frac{1}{80 \times 576}$$

which does the job. The number

$$1 - \frac{1}{4} + \frac{1}{48} - \frac{1}{1152}$$

yields the first three digits of L.

b) (10 points) Sum the series

$$\sum_{k=1}^{\infty} \frac{k}{2^k}$$

$$\sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{k}{2^{k-1}} = \frac{1}{2} \frac{1}{(1-1/2)^2} = 2 \ .$$

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III: a) (7 points) Write the power series expansion for

$$\log(1+x)\ .$$

$$\log(1+x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1}$$

b) (8 points) Write the power series expansion for

$$\log\left(\frac{1+x}{1-x}\right)$$

$$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1} + \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

c) (10 points) Write the power series expansion of

$$\int_0^x \frac{1}{1+t^4} dt\ .$$

$$\frac{1}{1+t^4} = \sum_{k=0}^{\infty}(-1)^k t^{4k}$$

and hence

$$\int_0^x \frac{1}{1+t^4} dt = \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k}}{4k+1}$$

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IV: a) (10 points) Find the general solution of the differential equation

$$y' + 2xy = x \ .$$

Check your answer!

Integrating factor is e^{x^2} and hence

$$\left(e^{x^2} y\right)' = e^{x^2} x$$

Thus

$$y(x) = \frac{1}{2} + Ce^{-x^2} \ .$$

b) (15 points) Solve the initial value problem

$$xy' + 4y = x^2 \quad , y(1) = 2 \ .$$

Check your answer!

Note that

$$x^4 y' + 4x^3 y = x^5$$

or

$$(x^4 y)' = x^5 \ .$$

Hence

$$y(x) = \frac{1}{6} x^2 + \frac{C}{x^4}$$

$$2 = y(1) = \frac{1}{6} + C$$

and hence

$$y(x) = \frac{1}{6}\left(x^2 + \frac{11}{x^4}\right) \ .$$