Test IV for Calculus II, Math 1502, November 10, 2009

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This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

Print your name (not your signature), your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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I: (10 points each) For the two systems of equations below, decide whether solutions exist or not. If they do exist calculate all of them.
a)

$$\begin{aligned} x + 2y &= 3 \\ 2x + 3y - z &= 4 \\ 2x + y - 3z &= 1 \end{aligned}$$

Row reduction leads to

$$\begin{bmatrix} 1 & 2 & 0 & | & 3 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$$

and there is no solution.
b)

$$\begin{aligned} x + 2y &= 3 \\ 2x + 3y - z &= 4 \\ 2x + y - 3z &= 0 \end{aligned}$$

Row reduction leads to

$$\begin{bmatrix} 1 & 2 & 0 & | & 3 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

and there are infinitely many solution given by

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

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II: Consider the matrix $A = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$ where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad , \quad \vec{v}_2 = \begin{bmatrix} -3 \\ -4 \\ -1 \end{bmatrix} \quad , \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad .$$

a) (5 points) Give a spanning set for $\text{Img}(A)$.

The three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ form a spanning set.

b) (5 points) Give a basis for $\text{Img}(A)$.

Row reduction of the matrix $[\vec{v}_1, \vec{v}_2, \vec{v}_3]$ leads to

$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad .$$

The first and second column are pivotal and hence $\vec{v}_1, \vec{v}_2$ form a basis for $\text{Img}(A)$.

c) (10 points) Find a basis for $\text{Ker}(A)$.

Using the row reduction in part b) on finds that

$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

is a basis for the $\text{Ker}(A)$.

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III: True or false: (5 points each)

a) For a symmetric matrix A, i.e., $A = A^T$, $\mathrm{Ker}(A)^\perp = \mathrm{Img}(A)$. TRUE

b) An $n \times m$ matrix whose rank is r has always $n - r$ non-pivotal columns. FALSE, it should read $m - r$ pivotal columns.

c) If the rank of a matrix A is r, so is the rank of the matrix A^T. TRUE

d) Four vectors in $\mathcal{R}^4$ are linearly independent if any three of them are. FALSE

e) An $n \times n$ matrix A is invertible if and only if $\mathrm{Ker}(A) = \{\vec{0}\}$. TRUE

IV: a) (15 points) Using the normal equations, solve the least square problem for $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 2 & -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{2}{15} \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

b) (5 points) Find the vector in $\mathrm{Img}(A)$ that is closest to the vector $\vec{b}$. This is the vector

$$A\vec{x} = \frac{2}{15} \begin{bmatrix} 13 \\ 11 \\ 5 \end{bmatrix}.$$

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V: A matrix A has the QR factorization with

$$Q = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 1 & -2 \end{bmatrix}, \quad \text{and} \quad R = \begin{bmatrix} 3 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

i) (10 points) Find the orthogonal projection onto $\mathrm{Img}(A)$.

$$P_{\mathrm{Img}(A)} = QQ^T = \frac{1}{9}\begin{bmatrix} 5 & 2 & 4 \\ 2 & 8 & -2 \\ 4 & -2 & 5 \end{bmatrix}$$

ii) (5 points) Find the orthogonal projection onto $\mathrm{Ker}(A^T)$.

$$P_{\mathrm{Ker}(A^T)} = I_{3\times 3} - QQ^T = \frac{1}{9}\begin{bmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 2 & -1 & -2 \end{bmatrix}.$$

iii) (Extra credit: 15 points) Find the least square solutions for $A\vec{x} = \vec{b}$ where $\vec{b} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$.

Solving the equation $R\vec{x} = Q^T\vec{b}$ yields

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$