Solutions to Practice Final for Calculus II, Math 1502, December 5, 2009

Name:

This test is to be taken without calculators and notes of any sorts. The allowed time is 2 hours and 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

Reminder: Final exam Dec 7th (Mon) 2:50pm - 5:40pm Howey (Physics) L3

Block 1:

1: Calculate to three digits accuracy

$$\int_1^2 e^{\frac{1}{x}} dx \ .$$

Solution: Recall

$$e^y = \sum_{k=0}^n \frac{y^k}{k!} + \frac{1}{n!}\int_0^y e^z (y-z)^n dz$$

and hence

$$\int_1^2 e^{\frac{1}{x}} dx = \sum_{k=0}^n \int_1^2 \frac{1}{x^k k!} dx + \frac{1}{n!}\int_1^2 \left[\int_0^{\frac{1}{x}} e^z (\frac{1}{x} - z)^n dz\right] dx \ .$$

We denote the last term in the formula as R_n, the remainder and estimate it now. Since

$$\frac{1}{n!}\int_0^{\frac{1}{x}} e^z (\frac{1}{x} - z)^n dz \le e^{\frac{1}{x}} \frac{1}{x^{n+1}(n+1)!}$$

and $1/x$ ranges between $1/2$ and 1 we find

$$R_n \le \int_1^2 e^{\frac{1}{x}} \frac{1}{x^{n+1}(n+1)!} dx \le e \int_1^2 \frac{1}{x^{n+1}(n+1)!} dx$$

$$\leq \frac{3}{n(n+1)!}(1-\frac{1}{2^n}) \leq \frac{3}{n(n+1)!} \ .$$

Now we choose n so that the last expression is $\leq 10^{-4}$. If we use $n = 5$ we get

$$R_n \leq \frac{1}{1200}$$

and using $n = 6$ we get

$$R_n = \frac{3}{6 \cdot 7!} = \frac{3}{30240} < 10^{-4} \ .$$

Hence the expression

$$\sum_{k=0}^{6} \int_1^2 \frac{1}{x^k k!} dx = \sum_{k=0}^{6} \frac{1}{k!} \int_1^2 \frac{1}{x^k} dx$$

$$= 1 + \log 2 + \sum_{k=2}^{6} \frac{1}{(k-1)k!}(1-\frac{1}{2^{k-1}}) \ .$$

yields the desired approximation.

2: a) Compute

$$\lim_{x \to 0} \frac{\ln(\cos x)}{x^2}$$

Solution: L'Hôpital's rule or Taylor's theorem yields

$$-\frac{1}{2} \ .$$

b) Calculate the integral provided it exists

$$\int_1^\infty \frac{\sin(\frac{1}{x})}{x^2} dx \ .$$

Solution: Consider first the integral

$$\int_1^L \frac{\sin(\frac{1}{x})}{x^2} dx$$

which, using the substitution $y = 1/x$ can be transformed into

$$\int_{\frac{1}{L}}^{1} \sin(y)dy = -\cos(y)\Big|_1^{\frac{1}{L}} = \cos(1) - \cos(\frac{1}{L}) \to \cos(1) - 1 \ ,$$

as $L \to \infty$. Hence the improper integral exists and equals

$$\cos(1) - 1$$

Block 2:

3: a) Find the interval of convergence (including the endpoints) of the power series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} (x-1)^k \ .$$

Solution: The series converges absolutely in $(0,2)$. At 2 it diverges. Note that $1/\sqrt{k}$ is positive decreasing. Hence at 0, being an alternating series, it converges.

b) Does the series

$$\sum_{k=1}^{\infty} (-1)^k \frac{\ln k}{k}$$

converge? If it does converge, give a reasonable estimate on n so that s_n, the n-th partial sum, and the limit differ by 10^{-5}.

Solution: The function

$$\frac{\ln x}{x}$$

has the derivative

$$\frac{1 - \ln x}{x^2}$$

and hence is a decreasing function for $x \geq e$. Moreover, the function tends to zero as $x \to \infty$. Hence the series converges and we find that the limit L satisfies

$$|L - s_n| \leq \frac{\ln(n+1)}{n+1}$$

Pick $n = 10^7 - 1$ and note that $\ln 10^7 = 7 \ln 10$. Now $e^3 > 10$ and hence $\ln 10^7 < 21$. Thus

$$\frac{\ln(10^7)}{10^7} < \frac{21}{10^7} < 10^{-5} \ .$$

4: Solve the differential equations

$$xy' + 5y = x^3 \ , \ y(1) = 1$$

and

$$y' = x(1 + y^2) \ , \ y(0) = 0 \ .$$

Solution: Integrating factor is x^4 and therefore multiplying the first differential equation by x^4 yields

$$x^7 = x^5 y' + 5x^4 y = (x^5 y)' \ ,$$

which upon integration yields

$$x^5 y = \frac{x^8}{8} + C \ ,$$

or

$$y(x) = \frac{x^3}{8} + \frac{C}{x^5} \ .$$

The initial condition determines $C = \frac{7}{8}$ and

$$y(x) = \frac{x^3}{8} + \frac{7}{8x^5} \ .$$

The second differential equation we solve by separation of variables

$$\frac{y'}{1 + y^2} = x \ ,$$

which can be written as

$$\frac{d}{dx} \tan^{-1}(y) = \frac{d}{dx} \frac{x^2}{2} \ .$$

This yields

$$\tan^{-1}(y) = \frac{x^2}{2} + C \; .$$

The initial condition requires $C = 0$ and hence

$$y(x) = \tan(\frac{x^2}{2}) \; .$$

Block 3:

5: Let f be a linear transformation from $\mathcal{R}^3$ to $\mathcal{R}^3$ such that

$$f\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\\1\end{bmatrix} \; , \; f\left(\begin{bmatrix}2\\1\\3\end{bmatrix}\right) = \begin{bmatrix}-1\\1\\-2\end{bmatrix}$$

and

$$f\left(\begin{bmatrix}3\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-1\\1\end{bmatrix} \; .$$

Find the matrix associated with f.

Solution: The matrix A_f associated with the linear transformation f must satisfy $A_f B = C$ where

$$B = \begin{bmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 3 & 1\end{bmatrix} \; , \; C = \begin{bmatrix}2 & -1 & 0\\1 & 1 & -1\\1 & -2 & 1\end{bmatrix} \; .$$

Using row reduction we calculate the inverse of B

$$B^{-1} = \frac{1}{2}\begin{bmatrix}-1 & -7 & 3\\0 & 2 & 0\\1 & 1 & -1\end{bmatrix}$$

and since $A_f = CB^{-1}$, we find

$$A_f = \begin{bmatrix}-1 & -8 & 3\\-1 & -3 & 2\\0 & -5 & 1\end{bmatrix} \; .$$

6: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 3 & 2 & 1 \end{bmatrix} .$$

Is there a vector $\vec{b} \in \mathcal{R}^3$ for which $A\vec{x} = \vec{b}$ has a solution? Either find it or explain why it does not exist.

Solution: Row reducing the augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & | & x \\ 4 & 4 & 4 & | & y \\ 3 & 2 & 1 & | & z \end{bmatrix}$$

leads to

$$\begin{bmatrix} 1 & 2 & 3 & | & x \\ 0 & -4 & -8 & | & y-4x \\ 0 & 0 & 0 & | & z+x-y \end{bmatrix} .$$

Thus the vector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

is in Img(A) if and only if

$$z + x - y = 0 .$$

Block 4:

7: Find the QR factorization of the matrix

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

and solve the least square problem $A\vec{x} = \vec{b}$ where

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} .$$

Is this least square solution unique? If not find the one that has least length.

Solution: The Gram-Schmidt procedure yields

$$Q = \frac{1}{3}\begin{bmatrix} 2 & \frac{-5}{\sqrt{5}} \\ 2 & \frac{4}{\sqrt{5}} \\ 1 & \frac{2}{\sqrt{5}} \end{bmatrix}$$

and since $R = Q^T A$

$$R = \frac{1}{3}\begin{bmatrix} 9 & 7 & 13 \\ 0 & \frac{5}{\sqrt{5}} & -\frac{10}{\sqrt{5}} \end{bmatrix}$$

To find the least square solution we calculate

$$Q^T\vec{b} = \frac{1}{3}\begin{bmatrix} 5 \\ \frac{1}{\sqrt{5}} \end{bmatrix} ,$$

and solve $R\vec{x} = Q^T\vec{b}$

$$\vec{x}(s) = \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} .$$

The least square solution is not unique.

To find the solution of least length, call it $\vec{x}_0$, there are two possibilities. The first is to calculate the square of the length of the vector $\vec{x}(s)$

$$(\frac{2}{5} - 3s)^2 + (\frac{1}{5} + 2s)^2 + s^2$$

minimizing this function with respect to s yields

$$s = \frac{2}{35}$$

and

$$\vec{x}_0 = \frac{1}{35}\begin{bmatrix} 8 \\ 11 \\ 2 \end{bmatrix} .$$

Another way of getting to this result is to split the vector

$$\vec{x}^0 := \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \\ 0 \end{bmatrix} .$$

into two components, one $\vec{x}^0_{||}$ parallel to the Ker(A) and the other $\vec{x}^0_{\perp}$ perpendicular to Ker(A). The vector $\vec{x}^0_{\perp}$ is the desired answer. Note that Ker(A) is one dimensional and is spanned by the vector

$$\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} .$$

The projection onto this one dimensional subspace is given by

$$P := \frac{1}{14} \begin{bmatrix} 9 & -6 & -3 \\ -6 & 4 & 2 \\ -3 & 2 & 1 \end{bmatrix}$$

and the projection onto the complement of Ker(A) is given by

$$I - P = \frac{1}{14} \begin{bmatrix} 5 & 6 & 3 \\ 6 & 10 & -2 \\ 3 & -2 & 13 \end{bmatrix}$$

and

$$\vec{x}^0_{\perp} = (I - P)\vec{x}^0 = \frac{1}{35} \begin{bmatrix} 8 \\ 11 \\ 2 \end{bmatrix} .$$

8: Find a basis for the kernel and the image for the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 2 \end{bmatrix}$$

Give an equation for the image as well. Also, find the orthogonal projections onto Img(A) and Ker(A).

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & x \\ 2 & 0 & 1 & y \\ 3 & 2 & 2 & z \end{array}\right]$$

which upon row reduction leads to

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & x \\ 0 & -4 & -1 & y-2x \\ 0 & 0 & 0 & z-x-y \end{array}\right] \qquad (*)$$

From this we see that a vector is in Img(A) if and only if it is in the plane $x+y-z=0$. A basis for this plane is given, e.g., by

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

An orthonormal basis for Img(A) can be found using the Gram-Schmidt procedure whidh yields

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

and if we set

$$Q = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

the orthogonal projection P onto Img(A) is given by

$$P = QQ^T = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

The kernel of the matrix A is one dimensional. Setting $x=y=z=0$ in (*) we can use backward substitution and find that Ker(A) is spanned by the normalized vector

$$\vec{u} = \frac{1}{\sqrt{21}} \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}.$$

The projection onto Ker(A) is then given by

$$\vec{u}\vec{u}^T = \frac{1}{21}\begin{bmatrix} 4 & 2 & -8 \\ 2 & 1 & -4 \\ -8 & -4 & 16 \end{bmatrix} .$$

Block 5:

9: Sketch the curve defined by the equation

$$10x^2 + 8xy + 4y^2 = 12 .$$

Solution: The equation can be written as $\vec{x} \cdot A\vec{x} = 12$ where

$$A = \begin{bmatrix} 10 & 4 \\ 4 & 4 \end{bmatrix} .$$

The characteristic polynomial s given by $t^2 - 14t + 24 = (t-12)(t-2)$. Thus, we have the eigenvalues 12 and 2 with the corresponding eigenvectors

$$\vec{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} , \vec{u}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

We can now diagonalize the matrix A and get $A = UDU^T$ where

$$U = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 12 & 0 \\ 0 & 2 \end{bmatrix} .$$

If we rewrite the equation in terms of the variables

$$U^T\vec{x} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} =: \vec{y}$$

we find

$$12\tilde{x}^2 + 2\tilde{y}^2 = 12 \text{ or } \tilde{x}^2 + \frac{\tilde{y}^2}{6} = 1 ,$$

which is an ellipse with the large semi axis of length $\sqrt{6}$ along the $\tilde{y}$ axis and the short semi axis of lenght 1 along the $\tilde{x}$ axis.

To se this in the $x - y$ picture, just draw the same figure but with the $\tilde{x}$ axis replaced by the axis along the vector $\vec{u}_1$ and the $\tilde{y}$ axis replaced by the axis along the vector $\vec{u}_2$.

10: Find the solution of the system of differential equations

$$x' = x + y \ , \ y' = -x + 3y$$

with the initial conditions $x(0) = 2$, $y(0) = 1$.

Solution: Write this system as $\vec{x}' = A\vec{x}$ where

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \ , \ \vec{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \ .$$

This matrix has the characteristic polynomial

$$t^2 - 4t + 4 = (t-2)^2 \ ,$$

and hence the eigenvalue 2 has algebraic multiplicity 2. Now we note that

$$N := A - 2I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

and N^2 is the zero matrix. Now

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

and since

$$A^k = 2^k I + k2^{k-1} N$$

we have that

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!}(2^k I + k2^{k-1}N) = \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} I + \sum_{k=0}^{\infty} \frac{k2^{k-1}t^k}{k!} N$$

$$= e^{2t} I + te^{2t} N \ .$$

Thus

$$e^{At} = e^{2t} \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} .$$

The solution is then

$$\vec{x}(t) = e^{At}\vec{x}(0) = e^{2t} \begin{bmatrix} 2-t \\ 1-t \end{bmatrix} .$$

11: Diagonalize the matrix

$$\begin{bmatrix} 17 & -2 & -2 \\ -2 & 14 & -4 \\ -2 & -4 & 14 \end{bmatrix}$$

Solution: The characteristic polynomial is

$$[(17-t)(10-t)-8](18-t)$$

with roots 18 and 9. The eigenvalue 9 has the eigenvector

$$\frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

and the eigenspace of the eigenvalue 18 is spanned by

$$\frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} , \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} .$$

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

12: Solve, i.e., calculate a_n for all n, the finite difference equation

$$a_{n+1} = 3a_n + 4a_{n-1}\ , n = 0, 1, 2, \ldots$$

with the initial condition $a_0 = 1, a_1 = 1$.

Solution: Set

$$\vec{x}_n = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$$

then

$$\vec{x}_{n+1} = A\vec{x}_n$$

where

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}\ .$$

$$\begin{aligned} A^n &= \frac{1}{5}\begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4^n & 0 \\ 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 4^{n+1} + (-1)^n & 4^{n+1} - 4(-1)^n \\ 4^n - (-1)^n & 4^n + 4(-1)^n \end{bmatrix}\ . \end{aligned}$$

Hence

$$a_n = \frac{2 \cdot 4^n - 3(-1)^{n-1}}{5}\ , n = 0, 1, 2, \cdots$$