Practice Test I A Solutions for Calculus II, Math 1502, September 1, 2009

Name:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

I: (25 points)

a) Find the 12-th order Taylor polynomial $P_{12}(x)$ for the function $\cos(x^3)$.

$$\cos y = 1 - \frac{y^2}{2} + \frac{y^4}{4!} - \frac{1}{5!}\int_0^y \cos z (y-z)^5 dz$$

and hence setting $y = x^3$ we get

$$P_{12}(x) = 1 - \frac{x^6}{2} + \frac{x^{12}}{4!} \ .$$

b) Using the above result, compute an approximate value for

$$\int_0^1 \frac{\cos(x^3) - 1}{x^3} dx \ .$$

This integral equals

$$\int_0^1 [-\frac{x^3}{2} + \frac{x^9}{4!}] dx = -\frac{1}{8} + \frac{1}{240} = \frac{-29}{240} \ .$$

c) Give an estimate on how accurate the value computed in b) approximates the integral.

The remainder is given by

$$-\int_0^1 \frac{1}{5! x^3} \int_0^{x^3} \cos z (x^3 - z)^5 dz dx$$

which is in magnitude less than

$$\int_0^1 \frac{1}{5!x^3} \int_0^{x^3} (x^3 - z)^5 dz dx = \int_0^1 \frac{1}{5!x^3} \frac{-(x^3 - z)^6}{6} |_0^{x^3} dx$$

$$= \int_0^1 \frac{x^{15}}{6!} dx = \frac{1}{16 \times 6!} = \frac{1}{11520} .$$

II: (25 points) Compute the limits:

a)

$$\lim_{a \to 0} \frac{a^x - 1 - \log(a)x}{x^2} = \frac{(\log a)^2}{2}$$

b)

$$\lim_{n \to \infty} (e^{2n} + e^n)^{1/n} = e^2$$

c)

$$\frac{\int_0^{\sin(x)} \sin(y^2) dy}{(e^x - e^{-x})^3} = \frac{1}{24}$$

III: (25 points) Decide which of the following improper integrals exists and compute its values if it exists:

a)

$$\int_0^2 \frac{1}{(1-x)^2} \mathrm{d}x$$

Does not exist. The problem is at $x = 1$

b)

$$\int_0^{1/2} \frac{\sin(x)}{x^2 \ln(x)} \mathrm{d}x$$

The sine function behaves as x for $x \to 0$ and hence the convergence issue is the same as for the integral

$$\int_0^{1/2} \frac{1}{x \ln(x)} \mathrm{d}x = \int_{-\infty}^{-\log 2} \frac{1}{t} dt$$

which does not converge.

c)

$$\int_0^\infty x e^{-x} \mathrm{d}x$$

This integral exists and equals to 1.

d) Extra credit:

$$\int_1^\infty \cos(x^2) dx$$

Set $x = \sqrt{s}$ and write the integral

$$\int_1^L \cos(x^2) dx = \frac{1}{2} \int_1^{\sqrt{L}} \cos(s) \frac{1}{\sqrt{s}} ds \ .$$

Integrating by parts yields

$$\int_1^L \cos(x^2) dx = \sin(s) \frac{1}{2\sqrt{s}} \Big|_1^{\sqrt{L}} + \frac{1}{4} \int_1^{\sqrt{L}} \sin(s) \frac{1}{s^{3/2}} ds$$

As L tends to infinity the first term converges to

$$-\frac{\sin(1)}{2}$$

whereas the second term converges since $1/s^{3/2}$ is integrable on $[1, \infty)$ using the comparison test.

IV: (25 points) Which of the following series is convergent or divergent.

a)

$$\sum_{k=2}^\infty \frac{1}{k \log k} \ .$$

Divergent since

$$\int_2^\infty \frac{1}{x\log(x)}dx$$

is divergent.

b)

$$\sum_{k=1}^\infty \frac{k^2}{k^3+k^2-1} \ .$$

Is divergent since the summands behave as $1/k$ for large k and $\sum 1/k$ does not converge.

c) Evaluate the series

$$\sum_{k=-3}^\infty \left(\frac{9}{10}\right)^k \ .$$

$$= \sum_{k=0}^\infty \left(\frac{9}{10}\right)^{k-3} = \left(\frac{9}{10}\right)^{-3} \sum_{k=0}^\infty \left(\frac{9}{10}\right)^k = \frac{10^4}{9^3}$$