Solutions for Practice Test II for Calculus II, Math 1502, September 21, 2009

Name:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

I: (25 points)

a) (7 points) Does the series

$$\sum_{k=2}^{\infty}(-1)^k \frac{1}{k \log k}$$

converge absolutely?

By the integral test we have to show the existence of the integral

$$\int_2^{\infty} \frac{1}{x \log x} dx = \int_{\log 2}^{\infty} \frac{1}{t} dt$$

which is not finite. Hence the series is not absolutely convergent. It is conditionally convergent since it is an alternating series and the sequence $1/(l \log k)$ is monotonically decreasing to zero as $k \to \infty$.

b) (10 points) Consider the series

$$\sum_{k=0}^{\infty}(-1)^k \frac{2^k}{(2k)!}$$

Does this series converge? If yes, calculate the first fife digits after the decimal point of this limit.

Yes, it is alternating and if L denotes the limit we have that

$$|L - s_n| \le \frac{2^{n+1}}{(2(n+1))!}$$

If we pick $n = 4$ we find that

$$\frac{2^{n+1}}{(2(n+1))!} = \frac{32}{10!} = \frac{32}{3,628,800} < 10^{-6} \ .$$

Hence the first five digits of the limit are given by

$$1 - \frac{2}{2!} + \frac{4}{4!} - \frac{8}{6!} + \frac{32}{8!} = \frac{1}{6} - \frac{1}{90} + \frac{1}{1260} \ .$$

c) (8 points) Let

$$\sum_k a_k x^k$$

be a power series and assume that it converges at $c > 0$. True or false:

1) The series converges for all $x < c$. NO
2) The radius of convergence, $r \geq c$. NO
3) The radius of convergence, $r \leq c$. NO
4) The series converges absolutely for x with $|x| < c$. YES

II: (25 points)

a) (8 points) Find the first three digits after the decimal point of

$$\int_0^1 \cos(x^2) dx \ .$$

$$\cos(y) = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!}$$

and hence

$$\int_0^1 \cos(x^2) dx = \sum_{k=0}^{\infty} (-1)^k \int_0^1 \frac{x^{4k}}{(2k)!} dx = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!(4k+1)} \ .$$

The series is alternating and hence

$$\left|\int_0^1 \cos(x^2)dx - s_n\right| \leq \frac{1}{(2(n+1))!(4n+5)}$$

If we choose $n = 2$ we get

$$\frac{1}{6! \times 13}$$

which is already close to $10,000$. So the first three terms

$$\sum_{k=0}^{2}(-1)^k \frac{1}{(2k)!(4k+1)} = 1 - \frac{1}{10} + \frac{1}{216}$$

yield the answer to three digits accuracy.

b) (10 points) Sum the series

$$\sum_{k=2}^{\infty} k^2 \frac{1}{3^k}$$

Recall that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{(1-x)} = \sum_{k=1}^{\infty} kx^{k-1}$$

and likewise

$$\frac{2}{(1-x)^3} = \frac{d^2}{dx^2}\frac{1}{(1-x)} = \sum_{k=2}^{\infty} k(k-1)x^{k-2}$$

From this we get that

$$\frac{2}{(1-x)^3} = \sum_{k=2}^{\infty} k^2 x^{k-2} - \sum_{k=2}^{\infty} kx^{k-2} = \frac{1}{x^2}\sum_{k=2}^{\infty} k^2 x^k - \frac{1}{x}\left(\sum_{k=1}^{\infty} kx^{k-1} - 1\right)$$

or

$$\frac{2}{(1-x)^3} = \frac{1}{x^2}\sum_{k=2}^{\infty} k^2 x^k - \frac{1}{x}\left(\frac{1}{(1-x)^2} - 1\right)$$

so that

$$\sum_{k=2}^{\infty} k^2 x^k = \frac{2x^2}{(1-x)^3} + x\left(\frac{1}{(1-x)^2} - 1\right) = \frac{4x^2 - 3x^3 + x^4}{(1-x)^3} .$$

c) (7 points) Find the power series for

$$\int_0^x \frac{\sin t}{t} dt$$

$$\sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)(2m+1)!}$$

III: (25 points)

a) (7 points) Find the Taylor series for the function

$$\log(1 - x^2)$$

$$-\sum_{k=1}^{\infty} \frac{x^{2k}}{k}$$

b) (10 points) Find the power series for $\tan^{-1}(x)$.

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2} dt = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

using that $\tan^{-1}(0) = 0$.

c) (8 points) Find the power series of $\sin(x)\cos(x)$.

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x) = \frac{1}{2}\sum_{k=0}^{\infty} (-1)^k \frac{(2x)^{2k+1}}{(2k+1)!}$$

IV: (25 points)

a) (7 points) Find the general solution of the differential equation

$$xy' + 2y = \frac{\cos x}{x}$$

$$y(x) = \frac{\sin x}{x^2} + \frac{c}{x^2}$$

b) (8 points) Solve the initial value problem

$$y' + xy = 1 \ , \ y(0) = 1 \ .$$

General solution:

$$y(x) = ce^{-x^2/2} + e^{-x^2/2} \int_0^x e^{y^2/2} dy$$

With the initial condition

$$y(x) = e^{-x^2/2} + e^{-x^2/2} \int_0^x e^{y^2/2} dy$$

c) (10 points) A 1000 gallon tank is full of brine with a concentration of 50 grams per gallon. The mixture is emptied at a rate of 5 gallons per minute and replenished (also at a rate of 5 gallons per minute) by a mixture that contains 30 grams per gallon. Find the amount of salt in the tank after t minutes.

The differential equation for the amount of salt at time t, $P(t)$ is given by

$$\frac{dP}{dt} = 150 - 5\frac{P}{1000}$$

since the amount of fluid in the tank stays constant. The amount of salt is measured in grams. Hence we have to solve

$$\frac{dP}{dt} + \frac{P}{200} = 150$$

and the general solution is given by

$$P(t) = 30,000 + Ce^{-\frac{t}{200}}$$

$$P(0) = 50,000$$

and hence

$$C = 20,000 \ ,$$

and

$$P(t) = 30,000 + 20,000e^{-\frac{t}{200}} \ .$$