Practice Test IV for Calculus II, Math 1502, November 4, 2009

Name:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

I: (15 points) The matrix A and the vector $\vec{B}$ are given by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 2 & 2 & a \end{bmatrix} \quad , \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ b \end{bmatrix} \ .$$

For which values of a, b is there no solution? For which values of a, b is there exactly one respectively infinitely many. Compute them them.

Solution: Row reduction of the augmented matrix leads to

$$\begin{bmatrix} 1 & 2 & 1 & |1 \\ 0 & 1 & 1 & |1 \\ 0 & 0 & a & |b \end{bmatrix}$$

There is no solution if $a = 0$ and $b \neq 0$. If $a \neq 0$ there is exactly one solution given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 + \frac{b}{a} \\ 1 - \frac{b}{a} \\ \frac{b}{a} \end{bmatrix} \ .$$

Finally, if $a = b = 0$ there are infinitely many solutions given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 + s \\ 1 - s \\ s \end{bmatrix} \quad , s \in \mathcal{R} \ .$$

II: (20 points) Using the normal equations, find the least square solutions of the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad , \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \ .$$

Solution: The normal equations are $A^T A\vec{x} = A^T\vec{b}$ which leads to

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} .$$

and

$$\vec{x} = \frac{1}{3}\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad , \quad s \in \mathcal{R} .$$

III: (15 points) Are the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad , \quad \vec{v}_2 = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \quad , \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

linearly independent?

Solution: Have to check whether $\mathrm{Ker}(A) = \{\vec{0}\}$ where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} .$$

Row reduction leads to

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & -1 & \frac{1}{9} \\ 0 & 0 & \frac{1}{14} - \frac{1}{9} \end{bmatrix} ,$$

which shows that all columns are pivotal. Hence $\mathrm{Ker}(A) = \{\vec{0}\}$ and the vectors are linearly independent.

IV: (20 points) The matrix A is given by

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 4 & 2 & 6 & 0 \\ 4 & 3 & 1 & 2 \\ 1 & 2 & 0 & 3 \end{bmatrix} .$$

Find a basis for Img(A) and for Ker(A).

Solution: Row reduction leads to

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

from which we glean that the first three column vectors are linearly independent and hence form a basis for Img(A). The vector

$$\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

spans the kernel and hence yields a basis for Ker(A). Note that

$$\dim \text{Img}(A) + \dim \text{Ker}(A) = 3 + 1 = 4 \ .$$

V: (15 points) Compute the inverse of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \ .$$

Solution:

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 4 & 2 & -3 \\ -2 & -1 & 4 \\ -5 & 0 & 5 \end{bmatrix} \ .$$

VI: a) (5 points) Given the basis in $\mathcal{R}^3$

$$\vec{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \ , \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \ , \quad \vec{u}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \ ,$$

calculate the components of the vector

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

in this basis.

Solution: Note that the vectors are orthonormal: Hence

$$\vec{b} := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3$$

where

$$x_1 = \vec{b} \cdot \vec{u}_1 = 2\sqrt{3}\ , x_2 = \vec{b} \cdot \vec{u}_2 = -\frac{1}{\sqrt{2}}\ , x_3 = \vec{b} \cdot \vec{u}_3 = -\sqrt{\frac{3}{2}}\ .$$

b) (10 points) The image of a $3 \times m$ matrix A is given by the equation $x + y + z = 0$. Find a basis for $\mathrm{Ker}(A^T)$.

Solution: It is a general fact that $\mathrm{Img}(A)$ consists of all the vectors in $\mathcal{R}^3$ that are perpendicular to $\mathrm{Ker}(A^T)$. In the problem at hand $\mathrm{Img}(A)$ consists of all vectors in $\mathcal{R}^3$ that are perpendicular to the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and hence the vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is a basis for $\mathrm{Ker}(A^T)$.