Practice Final Exam A for Math 1502, Calculus II

PART ONE – CORRESPONDING TO TEST ONE

(I): (15 points) Consider the function $f(y) = 1 - \cos(y)$. Let $R_n(y)$ be the remainder in the nth degree Taylor approximation for f; i.e.,

$$f(y) = P_n(y) + R_n(y) \ .$$

(a) Compute a close bound on

$$\left| \int_0^1 \frac{R_n(x^3)}{x} \mathrm{d}x \right| \ .$$

Your answer should be a function of n.

(b) Using a Taylor polynomial approximation, find an elementry integral that can be used to compute

$$\int_0^1 \frac{1 - \cos(x^3)}{x} \mathrm{d}x$$

to an accuracy of $\pm 10^{-3}$. Justify your answer.

(II): (5 points)
(a) Find the fifth degree Taylor polynomial $P_5(x)$ for

$$f(x) = \frac{1 - \cos(x^2)}{x^2}$$

about $x = 0$.

(b) Compute

$$\lim_{x \to 0} \frac{1 - \cos(x^2)}{\sin(x^2) x^2}$$

PART TWO – CORRESPONDING TO TEST TWO

(III): (10 points) Find the exact region of convergence of the following power series:

(a)

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k}(2+x)^k$$

(b)

$$\sum_{k=2}^{\infty} \frac{1}{\sqrt{k^2-1}} x^k$$

(IV): (10 points) For each of the following differnetial equations, find both the genral solution and the particular solution with $y(1) = 1$.

(a)

$$xy' + 4y = 2$$

(b)

$$y' = x^2 y$$

PART THREE – CORRESPONDING TO TEST THREE

(V): (10 points) Consider the system of equations

$$\begin{aligned} x + 2y + z &= b \\ 2x + y + 2z &= 2 \\ 3x + 3y + az &= 3 \end{aligned}$$

(a) For which values of a and b, if any, does this system have a unique solution? Give the solution for any such values of a and b.

(b) For which values of a and b, if any, does this system have no solution?

(c) For which values of a and b, if any, does this system have infinitely many solutions?

(VI): (10 points) Compute the inverse of

$$B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{pmatrix}$$

PART FOUR – CORRESPONDING TO TEST FOUR

(VII): (20 points) Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} .$$

(a) Find an orthonormal basis for the column space of A, and find a matrix Q with orthonormal columns and another matrix R so that

$$A = QR .$$

(b) Find an orthonormal basis for the row space of A, and find the orthogonal projection P_r onto the row space of A.

(c) Let $\mathbf{b}$ be the vector

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} .$$

Find a least squares solution to $A\mathbf{x} = \mathbf{b}$. Which vector in the column space of A is closest to $\mathbf{b}$?

(d) Give parametric descriptions of the column space of A and the kernel of A. If either one of these is a plane, give the eqaution of the plane.

(e) What is the dimension of the row space of A? What is the dimension of the column space of A?

(f) Are the rows of of A linearly independent? Are the columnss of of A linearly independent?

PART FIVE – CORRESPONDING TO MATERIAL AFTER TEST FOUR

(VIII): (20 points) Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} .$$

(a) Find the characteristic polynomial of A, and find all of its roots.

(b) Find the eigenvalues of A, and for each eigenvalue give a basis for the corresponding eigenspace.

(c) Find an orthogonal matrix U and a diagonal matrix D so that

$$A = UDU^{-1} .$$

(d) Compute $e^t A$.

(e) Solve the system of differential equations

$$\begin{aligned} x'(t) &= x(t) + y(t) \\ y'(t) &= x(t) + z(t) \\ z'(t) &= y(t) + z(t) \end{aligned}$$

with the inital conditions

$$x(0) = 1 \qquad y(0) = 2 \qquad z(0) = 1 .$$

(IX): (20 points) Consider the following matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1/2 \\ 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 \\ 3 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$$

(a) Which of these matrices, if any, have only a single eigenvalue, and which have two eigenvalues?

(b) Which of these matrices, if any, can be diagonalized by a change of basis?

(c) Compute $D^2 - 2D - 3I$ where I is the two by two identity matrix.

(d) Compute B^{15}