Practice Final Exam for Calculus II, Math 1502, December 10, 2010

Name:

Section:

Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 2 hours and 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given. **Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.**

Problem	Score
I	
II	
III	
IV	
V	
VI	
VII	
VIII	
IX	
X	
XI	
XII	
Total	

Name:

Section:

Name of TA:

Problems related to Block 1:

I: (15 points) Compute with an error less than 10^{-3}

$$\int_2^3 e^{\frac{1}{x^2}} dx \ .$$

$$e^y = \sum_{k=0}^n \frac{y^k}{k!} + \frac{e^c y^{n+1}}{(n+1)!}$$

where c is some number between 0 and y. Now we set $y = \frac{1}{x^2}$ and note that since x ranges between 2 and 3 the variable y ranges between $1/4$ and $1/9$. Hence we know that c can be a number that must be somewhere between 0 and $1/4$ and since e^y is monotone increasing we take $c = 1/4$ to obtain an upper bound on the remainder of the form

$$\frac{e^{1/4} y^{n+1}}{(n+1)!} \ .$$

Now from what we know about the exponential function $e < 3$ and hence $e^{1/4} < 3^{1/4}$ which is some number less than 2. Thus we find that

$$\begin{aligned} 0 &< \textstyle\int_2^3 \left[e^{\frac{1}{x^2}} - \sum_{k=0}^n \frac{x^{-2k}}{k!}\right] dx \le \frac{2}{(n+1)!} \int_2^3 x^{-2(n+1)} dx \\ &= \tfrac{2}{(n+1)!} \tfrac{1}{2n+1} \left[2^{-2n-1} - 3^{-2n-1}\right] < \tfrac{2}{(n+1)!} \tfrac{1}{2n+1} 2^{-2n-1} \ . \end{aligned}$$

If we choose $n = 3$ we find that the remainder of the integral is bounded by

$$\frac{2}{4!}\frac{1}{7} 2^{-7} = \frac{1}{84 \times 128} < \frac{1}{1000} \ .$$

Integrating the sum in the integral yields

$$\sum_{k=0}^3 \left[2^{-2k+1} - 3^{-2k+1}\right] \frac{1}{k!(2k-1)} \ .$$

II: a) (7 points) Compute the limit

$$\lim_{x \to 0} \frac{e^x - \cos x - \sin x}{x^3}$$

Using the Taylor expansion for all the functions in the numerator yields

$$e^x - \cos x - \sin x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots - 1 + \frac{x^2}{2} + \cdots - x - \frac{x^3}{3!} + \ldots$$

The leading order coefficient is x^2 and hence the limit does not exist.

b) (8 points) Does the improper integral

$$\int_0^1 \frac{1}{x^2} e^{\frac{1}{x}} dx$$

exist? If yes, compute it.

We have to compute

$$\lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{1}{x^2} e^{\frac{1}{x}} dx$$

and using the substitution

$$u = \frac{1}{x}$$

this integral can be rewritten as

$$\int_1^{\frac{1}{\varepsilon}} e^u du = e^{\frac{1}{\varepsilon}} - e^{-1}.$$

Clearly the limit as $\varepsilon \to 0$ does not exist.

Name:

Section:

Name of TA:

Problems related to Block 2:

III: a) (7 points) Is the series

$$\sum_{k=0}^{\infty}(-1)^k\frac{(k!)^2}{k^{2k}}$$

convergent? Is it absolutely convergent?

Using the ratio test we get that

$$\frac{((k+1)!)^2}{(k+1)^{2k+2}}\frac{k^{2k}}{(k!)^2}=\left(\frac{k}{k+1}\right)^{2k}$$

which converges to $\frac{1}{e^2}<1$. Hence the series is absolutely convergent and hence, in particular, convergent.

b) (8 points) Find the interval of convergence of the power series

$$\sum_{k=1}^{\infty}(-1)^k k^{-1+\frac{1}{k}}(x-2)^k$$

Set

$$a_k=\frac{|x-2|^k}{k^{1-\frac{1}{k}}}$$

and note that

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=|x-2|$$

hence the interval of convergence contains the interval $(1,3)$. Looking at the end point $x=3$ we find the series

$$\sum_{k=1}^{\infty}(-1)^k k^{-1+\frac{1}{k}}$$

which converges since it is alternating and the coefficients decrease monotonically to zero. Indeed the statement

$$(k+1)^{-1+\frac{1}{k+1}}<k^{-1+\frac{1}{k}}$$

is equivalent to the statement

$$(k+1)^{1-\frac{1}{k+1}}>k^{1-\frac{1}{k}}\ ,$$

or

$$(k+1)^{\frac{k}{k+1}}>k^{\frac{k-1}{k}}\ ,$$

Since $k \geq 1$ and since $\frac{k}{k+1} > \frac{k-1}{k}$ we have that

$$(k+1)^{\frac{k}{k+1}} > (k+1)^{\frac{k-1}{k}} > k^{\frac{k-1}{k}} \ .$$

Hence the series converges at $x = 3$. At $x = 1$, however, the series diverges, since

$$k^{-1+\frac{1}{k}} \times k \to 1$$

as $k \to \infty$ and since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, we also have that

$$\sum_{k=1}^{\infty} k^{-1+\frac{1}{k}}$$

diverges, by the limit comparison test.

IV: (15 points) Solve the initial value problem

$$y' - \frac{1}{x^2} y = e^{-\frac{1}{x}} \ , \ y(1) = \frac{2}{e} \ .$$

Multiply the equation by the integrating factor $e^{\frac{1}{x}}$

$$e^{\frac{1}{x}} y' - \frac{1}{x^2} e^{\frac{1}{x}} y = \left(e^{\frac{1}{x}} y \right)' = 1 \ .$$

Hence

$$y(x) = x e^{-\frac{1}{x}} + c e^{-\frac{1}{x}}$$

where c s a constant. $y(1) = \frac{2}{e}$ yields $c = 1$ and hence our solutions is

$$y(x) = (x+1) e^{-\frac{1}{x}} \ .$$

Name:

Section:

Name of TA:

Problems related to Block 3:

V: (20 points) Consider the system of equations

$$2x + y + z = b$$
$$x + y - 2z = 2$$
$$x - y + az = -1$$

Determine all values for a and b for which this system has a) non solution, b) exactly one solution, c) infinitely many solutions. In the case b) and c) Compute all the solutions in terms of a and b. The augmented matrix is

$$\begin{bmatrix} 2 & 1 & 1 & | & b \\ 1 & 1 & -2 & | & 2 \\ 1 & -1 & a & | & -1 \end{bmatrix}$$

Switching the first and second row leads to

$$\begin{bmatrix} 1 & 1 & -2 & | & 2 \\ 2 & 1 & 1 & | & b \\ 1 & -1 & a & | & -1 \end{bmatrix}$$

Row reduction leads to

$$\begin{bmatrix} 1 & 1 & -2 & | & 2 \\ 0 & -1 & 5 & | & b-4 \\ 0 & 0 & a-8 & | & 5-2b \end{bmatrix}$$

If $a = 8$ and $2b \neq 5$ there is no solution. If $a \neq 8$ there is always a unique solutions and if $a = 8$ and $2b = 5$ there are infinitely many solutions.

If $a \neq 8$ we can use back substitution and obtain:

$$z = \frac{5-2b}{a-8}\ ,\ y = 5\frac{5-2b}{a-8} + 4 - b\ ,\ x = -3\frac{5-2b}{a-8} + b - 2$$

If $a = 8$ and $2b = 5$ then the row reduced augmented matrix is

$$\begin{bmatrix} 1 & 1 & -2 & | & 2 \\ 0 & -1 & 5 & | & -\frac{3}{2} \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

and we find $z = t$, $y = 5t + \frac{3}{2}$ and $x = -3t + \frac{1}{2}$.

VI: (15 points) A plane in $\mathbb{R}^3$ passes through the points

$$\mathbf{p_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \ , \ \mathbf{p_2} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \ , \ \mathbf{p_3} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Give two representations of the plane, on in terms of parametrization and one in terms of an equation.

The plane is spanned by the vectors $\vec{v}_1 = \mathbf{p_2} - \mathbf{p_1}$, $\vec{v}_2 = \mathbf{p_3} - \mathbf{p_1}$ and passes through the point $\mathbf{p_1}$. Hence it is given by the parametrization

$$\vec{x}(s,t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

To find the equation we have to find a vector that is perpendicular to both vectors $\vec{v}_1$ and $\vec{v}_2$ and inspection shows that the vector $(1, 0, -1)$ does the job. We have to make sure that the vector $(1, 1, 1)$ has its tip on the plane. Hence the equation is given by

$$x - z = 0 \ .$$

Name:

Section:

Name of TA:

VII: (20 points) Use the least square method to find the distance of the tip of the vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

to the plane given by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solve the problem in two ways, once using the normal equations and then using the QR factorization.

If we set

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$$

then we have to solve the least square problem $A\vec{x} = \vec{b}$ where

$$\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} .$$

The normal equations are $A^T A\vec{x} = A^T\vec{b}$ and hence we have to solve

$$\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

which yields $x = -2/7$, $y = 6/7$. Thus, the vector in $Img(A)$ that is closest to $\vec{b}$ is given by

$$A\vec{x} = \frac{2}{7} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} .$$

For the distance we have to calculate the length of

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{7}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix},$$

given by

$$\sqrt{\frac{2}{7}}\,.$$

Now we use the QR factorization. We have to find an orthonormal basis for $Img(A)$. One vector is

$$\frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

and the other one is then found by looking at the vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

which normalized equals

$$\frac{1}{\sqrt{21}}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Hence Q is given by

$$Q = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{21}} \\ \frac{-1}{\sqrt{6}} & \frac{4}{\sqrt{21}} \end{bmatrix}$$

and hence with $Q^T A = R$ we get

$$\begin{bmatrix} \frac{6}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{7}{\sqrt{21}} \end{bmatrix}.$$

Now $R\vec{x} = Q^T\vec{b}$ leads to $x = -2/7, y = 6/7$ which checks. However, in order to compute the shortest distance the matrix R is not important. The projection of $\vec{b}$ onto $Img(A)$ is given by

$$QQ^T\vec{b} = Q\begin{bmatrix} 0 \\ \frac{6}{\sqrt{21}} \end{bmatrix} = \frac{2}{7}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

which checks with what we had before, and the distance vector from the tip of the vector $\vec{b}$ to $Img(A)$ is given by

$$\vec{b} - QQ^T\vec{b} = \frac{1}{7}\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} ,$$

as we had before. Of course the distance is then the same number as before.

VIII: (15 points) Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 5 & 6 \\ 1 & 0 & 1 & 3 \\ 4 & 1 & 5 & 12 \\ 2 & 1 & 4 & 7 \end{bmatrix}$$

Find a basis for $Img(A)$ and for $Ker(A)$ as well as for $Img(A^T)$ and for $Ker(A^T)$. Try do this with a little computation as possible. Row reducing A yields

$$\begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} ,$$

as a basis vector for the kernel of A. Thus we can say that the dimension of $Img(A)$ which is the same as the dimension of $Img(A^T)$ equals 3. Thus $Ker(A^T)$ is one dimensional. The row reduced matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $Img(A^T)$ is the orthogonal complement we have to find three linearly independent vectors that are perpendicular to the above vector. Thus we have to solve

$$-2w + x - y + z = 0$$

which leads to the one-one parametrization $z = t, y = s, x = r$ and $w = \frac{1}{2}[r - s + t]$. Hence we get

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

as a basis for $Img(A^T)$. We know that the first three columns of the matrix are pivotal columns and hence we have that the first three vectors from a basis for $Img(A)$. To find a basis for $Ker(A^T)$ we have to find a vector perpendicular to the first three column vectors which can be found by row reducing the system

$$\begin{bmatrix} 2 & 1 & 4 & 2 \\ 3 & 0 & 1 & 1 \\ 5 & 1 & 5 & 4 \end{bmatrix}$$

which yields

$$\begin{bmatrix} 2 & 1 & 4 & 2 \\ 0 & 3 & 10 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

and from which we get that the basis for the kernel of A^T is

$$\begin{bmatrix} 1 \\ 10 \\ -3 \\ 0 \end{bmatrix}$$

Name:

Section:

Name of TA:

IX: (15 points) Graph the curve given by the equation

$$11x^2 - 6xy + 19y^2 = 10 \ .$$

The associated matrix is given by

$$\begin{bmatrix} 11 & -3 \\ -3 & 19 \end{bmatrix}$$

whose characteristic polynomial is $\mu^2 - 30\mu + 20 = (\mu - 10)(\mu - 20)$. Hence the eigenvalues are $\mu_1 = 10, \mu_2 = 20$. The associated eigenvectors are

$$\vec{u}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and

$$\vec{u}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \ .$$

In the $u - v$ plane the cureve is given by

$$u^2 + 2v^2 = 1$$

which is an ellipse whose semiaxis in the u–direction has length 1 and whose semiaxis in the direction v has length $1/\sqrt{2}$. To get the picture in the $x - -y$ plane we have to rotate the $u - -v$ picture by the rotation ,atrix

$$U = [\vec{u}_1, \vec{u}_2] = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \ .$$

X: (15 points) Diagonalize the matrices

$$a) \quad \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \ .$$

The eigenvalue 6 has the eigenvector

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Perpendicular to this is the vector

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

which is an eigenvector with eigenvalue 3. Thus the vector

$$\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

which is perpendicular to both must be an eigenvector, since the matrix is symmetric. The associated eigenvalue is 3 also.

$$b) \quad \begin{bmatrix} 6 & 9 \\ 4 & 11 \end{bmatrix}$$

The characteristic polynomial is $\mu^2 - 17\mu + 30 = (\mu - 15)(\mu - 2)$. The iegenvector associated with $\mu_1 = 15$ is the vector

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the one associated to $\mu_2 = 2$ is

$$\begin{bmatrix} 9 \\ -4 \end{bmatrix}$$

Name:

Section:

Name of TA:

XI: (20 points) Solve the initial value problem given by the system

$$
\begin{aligned}
x' &= 8x + 9y \\
y' &= 4x + 13y \\
x(0) &= 1\ ,\ y(0) = 2
\end{aligned}
\tag{0.1}
$$

Use both methods, the superposition principle and the exponential of a matrix.

The eigenvalues and the corresponding eigenvectors are

$$\mu_1 = 17\ ,\ \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\mu_2 = 4\ ,\ \vec{u}_2 = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$$

The general solution is then given by

$$ae^{17t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + be^{4t} \begin{bmatrix} 9 \\ -4 \end{bmatrix}$$

Using the intial conditions we arrive at

$$\frac{22}{13} e^{17t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{13} e^{4t} \begin{bmatrix} 9 \\ -4 \end{bmatrix}\ .$$

XII: (15 points) Solve the recursive relation, i.e., find a_n for arbitrary values of n,

$$a_{n+1} = 8a_n + 9a_{n-1}$$

with $a_0 = a_1 = 1$.

Writing

$$\vec{x}_n = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$$

we can write the recursion as

$$\vec{x}_{n+1} = A\vec{x}_n$$

where

$$A = \begin{bmatrix} 8 & 9 \\ 1 & 0 \end{bmatrix} .$$

The solution can be gotten via

$$\vec{x}_n = A^{n-1}\vec{x}_1 .$$

A is diagonalized by

$$A = \begin{bmatrix} 9 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{10} \begin{bmatrix} 1 & 1 \\ -1 & 9 \end{bmatrix} .$$

Hence

$$A^{n-1} = \begin{bmatrix} 9 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9^{n-1} & 0 \\ 0 & (-1)^{n-1} \end{bmatrix} \frac{1}{10} \begin{bmatrix} 1 & 1 \\ -1 & 9 \end{bmatrix}$$

Applying this to the initial condition $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we find that

$$a_n = \frac{1}{5}\left(9^n + 4(-1)^n\right) .$$