Practice Test 3A for Calculus II, Math 1502, October 18, 2010

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Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.
Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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I: (25 points) Let $f : \mathcal{R}^2 \rightarrow \mathcal{R}^3$ be a linear transformation such that

$$f\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad f\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\2\end{bmatrix}$$

Find the matrix A_f associated with f.
The matrix A_f is given by

$$A_f = \left[f\left(\begin{bmatrix}1\\0\end{bmatrix}\right), \; f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right].$$

Since f is linear

$$f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = f\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

and

$$f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = f\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\2\end{bmatrix}$$

or

$$A_f \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}1 & 1\\2 & -1\\3 & 2\end{bmatrix}$$

or

$$A_f = \begin{bmatrix}1 & 1\\2 & -1\\3 & 2\end{bmatrix} \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\1 & 3\\5 & 1\end{bmatrix}$$

Please make sure that you understand the last step!

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II: (25 points) a) Given two vectors

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \ , \ \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix} \ .$$

Find the component $\vec{x}_{||}$ of $\vec{x}$ parallel to $\vec{v}$ and the component $\vec{x}_{\perp}$ of $\vec{x}$ perpendicular to $\vec{v}$. Check your answer.

First, we calculate the unit vector $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$

$$\vec{u} = \frac{1}{5} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix} \ .$$

$$\vec{x}_{||} = (\vec{x} \cdot \vec{u})\vec{u} = \frac{27}{25} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix} \ ,$$

$$\vec{x}_{\perp} = \vec{x} - \vec{x}_{||} = \frac{1}{25} \begin{bmatrix} 25 \\ 50 \\ 75 \\ 100 \end{bmatrix} - \frac{1}{25} \begin{bmatrix} 27 \\ 54 \\ 54 \\ 108 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} -2 \\ -4 \\ 21 \\ -8 \end{bmatrix}$$

b) Find the distance between the tip of $\vec{x}$ and the line that passes through the origin and has direction $\vec{v}$.

The distance is $|\vec{x}_{\perp}|$ which equals

$$\frac{1}{25}\sqrt{4 + 16 + 441 + 64} = \frac{1}{25}\sqrt{525} = \frac{\sqrt{21}}{5}$$

c) Find the angle between the vector $\vec{x}$ and the vector $\vec{v}$.

We have that

$$\cos\theta = \frac{\vec{x}\cdot\vec{v}}{|\vec{x}||\vec{v}|} = \frac{27}{\sqrt{30}\cdot 5}$$

and

$$\theta = \cos^{-1}\left(\frac{27}{\sqrt{30}\cdot 5}\right)$$

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III: (25 points) a) Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} .$$

Check your answer!

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & -2 & 0 \\ \frac{-1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

The problem is reduced to computing the inverse of the 2×2 matrix $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ and the 1×1 matrix 3

b) The unit cube is panned by the vector $\vec{e}_1, \vec{e}_2, \vec{e}_3$. Find the volume of the image of this unit cube under the matrix A.

Note that the image of the cube is given by a cylindrical figure with a parallelogram as base and with height 3. Hence the volume of the image is the determinant of the 2×2 matrix multimplied by 3 times the volume of the original cube which is 1. This yields 6 units for the volume.

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IV: (25 points) a) Find the plane in parametrized form that passes through points

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} , \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} , \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

The vector $\vec{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has its tip on the plane and the difference vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Hence the parametrization of the plane is given by

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} .$$

b) Find the equation for the plane.

For the equation we have to find $\vec{a}$ and d such that

$$\vec{a} \cdot \vec{x}_0 = d$$

and

$$\vec{a} \cdot \vec{v}_1 = \vec{a} \cdot \vec{v}_2 = 0$$

Hence we get the two equations for $\vec{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$-3b = 3c = 0$$

and hence we may choose $a = 1$ and then get

$$d = 1 \ .$$

Thus the equation is

$$x = 1 \ .$$