

Practice Test 4A Solutions for Calculus II, Math 1502, November 11, 2010**Name:****Section:****Name of TA:**

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write 1.414.... Show your work, otherwise credit cannot be given.

Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

Problem	Score
I	
II	
III	
IV	
Total	

Name:

Section:

Name of TA:

I: (25 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & -12 \\ 1 & 3 & -7 \end{bmatrix}.$$

a) Find the QR factorization of this matrix.

First we perform the Gram-Schmidt orthogonalization:

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Then

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix}$$

and

$$\vec{u}_2 = \frac{1}{\sqrt{82}} \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix}.$$

Now

$$\vec{w}_3 = \begin{bmatrix} 2 \\ -12 \\ -7 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{123}{82} \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix} = \frac{1}{82} \left(\begin{bmatrix} 164 \\ -984 \\ -574 \end{bmatrix} + \begin{bmatrix} 205 \\ 0 \\ 205 \end{bmatrix} + \begin{bmatrix} -369 \\ 984 \\ 369 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus

$$Q = \frac{1}{\sqrt{82}} \begin{bmatrix} \sqrt{41} & -3 \\ 0 & 8 \\ \sqrt{41} & 3 \end{bmatrix}$$

Hence

$$R = Q^T A = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 3 & -5 \\ 0 & \sqrt{41} & -3\sqrt{41} \end{bmatrix}$$

Let

$$\vec{b} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} .$$

b) Is $\vec{b} \in \text{Img}(A)$? No. Row reduce

$$\begin{bmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 4 & -12 & | & 4 \\ 1 & 3 & -7 & | & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 1 & -3 & | & 1 \\ 0 & 0 & 0 & | & -2 \end{bmatrix}$$

c) If not compute all the least square solutions of “ $A\vec{x} = \vec{b}$ ”. Have to solve

$$R\vec{x} = Q^T \vec{b} .$$

$$Q^T \vec{b} = \frac{1}{\sqrt{82}} \begin{bmatrix} 3\sqrt{41} \\ 35 \end{bmatrix}$$

This amounts to solving

$$2x + 3y - 5z = 3 , \quad y - 3z = \frac{35}{41}$$

The solutions are

$$x = \frac{9}{41} - 2t , \quad y = \frac{35}{41} + 3t , \quad z = t .$$

Name:

Section:

Name of TA:

II: (25 points) Consider again the matrix A above.

a) Find a basis for $Ker(A)$ and $Img(A)$. What is the dimension of $Ker(A)$ what is the dimension of $Img(A)$?

Apply row reduction:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & -12 \\ 1 & 3 & -7 \end{bmatrix} .$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & -12 \\ 0 & 3 & -9 \end{bmatrix} .$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} .$$

The first two columns are pivotal and hence

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} , \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} .$$

forms a basis for $Img(A)$. Thus $\dim(Img(A)) = 2$. Hence $\dim(Ker(A)) = 1$. To find a basis for the kernel we solve the equation $A\vec{x} = \vec{0}$. Since we have row reduced the matrix we get

$$z = t , y = 3t , x = -2t .$$

Hence the vector

$$\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

is a basis for $Ker(A)$.

b) Find an equation for $Img(A)$. The equation must be of the form $ax + by + cz = 0$, since $Img(A)$ is a subspace and hence must pass through the origin. The basis vectors must be in the plane and hence we have

$$a + c = 0, 4b + 3c = 0$$

from which we get

$$a = -c, b = -\frac{3}{4}c$$

and hence the equation is given by

$$-4x - 3y + 4z = 0.$$

c) Find an equation for $Img(A^T)$. Find a basis for $Ker(A^T)$.

We now that $Img(A)$ is given by all vectors that are perpendicular to the vector

$$\begin{bmatrix} -4 \\ -3 \\ 4 \end{bmatrix}.$$

We know from general principles that $(Img(A))^\perp = Ker(A^T)$. Thus the kernel of A^T consists of all vectors that are perpendicular to all the vectors in $Img(A)$. Hence the vector

$$\begin{bmatrix} -4 \\ -3 \\ 4 \end{bmatrix}.$$

is a basis for $Ker(A)$. Recall, that the dimensions of $Img(A)$ and $Img(A^T)$ are the same and hence the dimension of $Ker(A^T)$ is one, which checks.

We now that

$$\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

is a basis for $Ker(A)$. The space $Img(A^T)$ consists of all vectors that are perpendicular to all the vectors in $Ker(A)$. Hence for every vector in $Img(A^T)$ of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

we must have that

$$-2x + 3y + z = 0$$

It is an easy check that the column vectors of A^T , i.e., the row vectors of A satisfy this equation.

Name:

Section:

Name of TA:

III: (25 points) Consider all vectors $\vec{x} \in \mathbb{R}^4$ of the form

$$\vec{x} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

that satisfy the equation $w + 2x - y + z = 0$.

a) Is this set of vectors a subspace of \mathbb{R}^4 ?

The answer is ‘yes’. If

$$\vec{x}_1 = \begin{bmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

are two vectors in this set then $w_1 + 2x_1 - y_1 + z_1 = 0$ as well as $w_2 + 2x_2 - y_2 + z_2 = 0$.

Hence by adding the equations we see that

$$\vec{x}_1 + \vec{x}_2 = \begin{bmatrix} w_1 + w_2 \\ x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

satisfies

$$(w_1 + w_2) + 2(x_1 + x_2) - (y_1 + y_2) + (z_1 + z_2) = 0 .$$

Hence $\vec{x}_1 + \vec{x}_2$ is in this set. Likewise for any vector

$$\vec{x} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

in this set and any number $a \in \mathbb{R}$ we have that

$$(aw) + 2(ax) - (ay) + (az) = a(w + 2x - y + z) = 0 .$$

b) If the answer to a) is yes, find a basis for this subspace S .

To find the basis we have to row reduce the matrix

$$B = \begin{bmatrix} 1 & 2 & -1 & 1 \end{bmatrix}$$

The solutions are given by

$$z = t_3, y = t_2, x = t_1, w = -2t_1 + t_2 - t_3 .$$

Hence, a one to one parametrization of the solutions is given by

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t_1 + t_2 - t_3 \\ t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

and the vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for this subspace. To see that it is a basis one does either a computation or one notes that $Img(B)$ is one dimensional. Hence $Ker(B)$ which is our subspace is three dimensional. Since the three vectors span this space, they must form a basis.

c) Find a basis for the orthogonal complement of S .

The orthogonal complement must have dimension one and a basis is given by the vector

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}.$$

Name:

Section:

Name of TA:

IV: (25 points) Consider the plane in \mathbb{R}^3 given by

$$x + 2y + 2z = 3$$

and the vector

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

a) Is this plane a subspace of \mathbb{R}^3 ?

No, this is not a subspace. If we add the coordinates of two vectors in this plane and plug it into the equation we obtain 6 on the right side and not 3.

Find the distance of the tip of the vector \vec{b} to the plane. Use two approaches.

b) Use a geometric approach by finding the vector normal to the plane and then finding the point on the plane whose distance to the tip of \vec{b} is shortest.

The unit vector

$$\vec{n} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

is normal to the plane. The distance of the plane from the origin is the length of the vector that is perpendicular to the plane and has its tip on the plane. Thus it is of the form $a\vec{n}$ and plugging this into the equation we find that $a = 1$ and the distance is 1. Next split the vector \vec{b} in a component parallel to \vec{n} and perpendicular to \vec{n} . The parallel component is given by

$$\frac{5}{3}\vec{n}$$

and the distance is given by the length of this vector minus 1, i.e., $2/3$.

c) As a second approach formulate this problem as a least square problem?

We have to consider

$$|\vec{b} - \vec{x}|^2 = (1 - x)^2 + (1 - y)^2 + (1 - z)^2$$

for all x, y, z with $x + 2y + 2z = 3$ and make this expression as small as possible. Eliminating x this is the same problem as making

$$(1 - [3 - 2y - 2z])^2 + (1 - y)^2 + (1 - z)^2 = (2 - 2y - 2z)^2 + (1 - y)^2 + (1 - z)^2$$

as small as possible. In other words, we have to find the least square solution of the problem $A\vec{x} = \vec{c}$ where

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} y \\ z \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Using the normal equations $A^T A \vec{x} = A^T \vec{c}$ we compute

$$A^T A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix},$$

and

$$A^T \vec{c} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

Thus, the normal equations are

$$5y + 4z = 5, \quad 4y + 5z = 5,$$

which leads to $y = z = \frac{5}{9}$. Finally

$$(2 - \frac{10}{9} - \frac{10}{9})^2 + (1 - \frac{5}{9})^2 + (1 - \frac{5}{9})^2 = \frac{4 + 16 + 16}{9^2} = \left(\frac{2}{3}\right)^2$$

and the distance is $\frac{2}{3}$.