Practice Test 4B for Calculus II, Math 1502, November 15, 2010

Name:

Section:

Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.
Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

Problem	Score
I	
II	
III	
IV	
Total	

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I: (25 points) a) The kernel of an $m \times n$ matrix A has the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

as a basis.

a) (2 points) What is n? $n = 3$ since the kernel of A a subspace of $\mathbb{R}^3$.

b) (3 points) What is the dimension of $Img(A)$, $Img(A^T)$? $dim(Img(A)) = 1$, since $dim(Ker(A)) + dim(Img(A)) = 3$. It is always the case that $dim(Img(A)) = dim(Img(A^T))$ and hence $dim(Img(A^T)) = 1$.

c) (8 points) Find a basis for $Img(A^T)$. We use that $Img(A^T)) = Ker(A)^{\perp}$. Hence we have to find all the vectors in $\mathbb{R}^3$ that are perpendicular to both of the above vectors. This amounts to solving the equations

$$x + y + z = 0 \ , \ 2x + y + z = 0$$

which yields $x = 0$ and $y = -z$. Hence the solutions are

$$t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

and the vector

$$\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

is a basis for $Img(A^T)$.

d) (12 points) Find the orthogonal projection onto $Ker(A)$ and $Img(A^T)$. It is easiest to first compute the orthogonal projection onto $Img(A^T)$. If $\vec{x} \in \mathbb{R}^3$ is an arbitrary vector we find that its projection onto $Img(A^T)$ is

$$(\vec{u} \cdot \vec{x})\vec{u} \ ,$$

where $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$. In terms of a matrix this projection is

$$\vec{u}\vec{u}^T = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = P \ .$$

The projection onto the kernel is then given by

$$Q = I - P = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \ .$$

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II: Find the distance between the lines

$$\vec{x}_1(s) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} , \quad \vec{x}_2(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} .$$

a) (5 points) Write this problem as a least square problem.

$$\vec{x}_1(s) - \vec{x}_2(t) = \begin{bmatrix} 1 & -1 \\ 1 & -2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} - \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} .$$

Hence we have to solve the least square problem

$$\begin{bmatrix} 1 & -1 \\ 1 & -2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} .$$

b) (10 points) Use the QR factorization method to find the solution.

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} ,$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$$

and hence

$$\vec{u}_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix} .$$

Therefore

$$Q = \frac{1}{3\sqrt{2}} \begin{bmatrix} 3 & 1 \\ 3 & -1 \\ 0 & -4 \end{bmatrix}$$

and

$$R = Q^T A = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 \\ 0 & 3 \end{bmatrix} .$$

Further,

$$Q^T \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 3 \end{bmatrix} .$$

We have to solve the equation

$$2s - 3t = -1 \; , 3t = 3$$

and hence the solutions are

$$s = t = 1 \; .$$

The distance vector is now given by

$$\vec{x}_1(1) - \vec{x}_2(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ,$$

i.e., the lines intersect!

c) (10 points) Solve the normal equation to check your solution found in b). This is now not necessary, since it is easy to check that the lines intersect.

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III: a) (7 points) Are the following vectors linearly independent in $\mathbb{R}^4$?

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix} \begin{bmatrix} -4 \\ 7 \\ 18 \\ -1 \end{bmatrix}$$

Row reduction leads to

$$\begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the vectors are not linearly independent. In fact the last vector is given by by 2 times the first minus 3 times the second. The first two columns are pivotal and the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$$

form a basis for S. b) (3 points) What is the dimension of the space they span? Two dimensions.

c) (7 points) Find the orthogonal complement of the subspace S of $\mathbb{R}^4$ that is spanned by those vectors. Finding the orthogonal complement means finding a basis for it.

We have to solve the equations

$$w + 2x + 3y + 4z = 0 \ , \ 2w - x - 4y + 3z = 0$$

Using row reduction one easily finds that

$$\begin{bmatrix} -3 \\ -4 \\ 1 \\ 2 \end{bmatrix} , \begin{bmatrix} -4 \\ 3 \\ -2 \\ 1 \end{bmatrix} .$$

is a basis for $S^{\perp}$.

d) (8 points) Find the orthogonal projection onto S and $S^{\perp}$. The first two columns are pivotal and the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$$

form a basis for S. Note that they are orthogonal and hence

$$\vec{u}_1 = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$$

form an orthonormal basis for S. We set

$$Q = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 3 & -4 \\ 4 & 3 \end{bmatrix}$$

and find for the projection onto S

$$P_S = QQ^T = \frac{1}{6} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ -1 & 2 & 5 & 0 \\ 2 & 1 & 0 & 5 \end{bmatrix}$$

Now,

$$P_{S^{\perp}} = I - P_S = \frac{1}{6} \begin{bmatrix} 5 & 0 & 1 & -2 \\ 0 & 5 & -2 & -1 \\ 1 & -2 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{bmatrix}.$$

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IV: a) (13 points) Find the QR factorization of the matrix A where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ -2 & 0 & 2 \end{bmatrix} .$$

We perform the Gram-Schmidt procedure:

$$\vec{u}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \frac{2}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

and hence

$$\vec{u}_2 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} .$$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} .$$

Hence

$$Q = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 2 \end{bmatrix} ,$$

and

$$R = Q^T A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \end{bmatrix} .$$

b) (12 points) Find the orthogonal projection onto $Img(A)$. This projection is given by

$$QQ^T = \frac{1}{9} = \frac{1}{9} \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & -2 \\ 2 & -2 & 8 \end{bmatrix} .$$