Topics for Test 4

You should be familiar with the following concepts:

Subspace S of $\mathbb{R}^n$, which is a subset of $\mathbb{R}^n$ with the property that with any two vectors $\vec{x}, \vec{y} \in S$, $\vec{x} + \vec{y} \in S$ and for all $a \in \mathbb{R}$ and all $\vec{x} \in S$, $a\vec{x} \in S$. Important examples are the kernel of an $m \times n$ matrix A,i.e., $Ker(A) \subset \mathbb{R}^n$ and $Img(A) \subset \mathbb{R}^m$, the image of an $m \times n$ matrix A.

A **spanning set** of a subspace $S \subset \mathbb{R}^n$, which is a collection of vectors so that every vector in S can be written as a linear combination of them.

A collection of vectors is **linearly independent** of no vector of this collection can be written as a linear combination of the others. Alternatively, this means that the matrix A which has those vectors as columns has a kernel $Ker(A)$ that consists only of the zero vector.

A **basis** of a subspace S is a collection of vectors that spans S and is linearly independent. Every basis of the subspace S has the same number of vectors and this number is called the **dimension** of S.

For an $m \times n$ matrix A there are is the important dimension formula

$$dim(Ker(A)) + dim(Img(A)) = n$$

If S is a subspace of $\mathbb{R}^n$ then the **orthogonal complement** of S, whcih is denoted by $S^\perp$ consists of all vectors that are perpendicular to every vector in S. The important theorem here is that

$$[S^\perp]^\perp = S \ .$$

If A is an $m \times n$ matrix then

$$Ker(A) \oplus Img(A^T) = \mathbb{R}^n$$

$$Ker(A^T) \oplus Img(A) = \mathbb{R}^m$$

The meaning of these formulas is that

$$Ker(A)^{\perp} = Img(A^T)$$

both are subspaces of $\mathbb{R}^n$. Likewise,

$$Img(A)^{\perp} = Ker(A^T) \ .$$

An $n \times n$ matrix whose **kernel consists only of the zero vector is invertible.**

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The above concepts have a computational side to them.

Row reduction leads you to see the pivotal columns and the non-pivotal columns. For an $m \times n$ matrix A, the pivotal columns are a basis for $Img(A)$. The number $r(A)$ of those columns, is called the **rank of the matrix** A, which equals to the dimension of the image of A, i.e.,

$$dim(Img(A)) = r(A) \ .$$

The number of non-pivotal columns determines the number of free variables which is the same as $dim(Ker(A))$.

You can check whether the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent by computing the kernel of the matrix $A = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k]$. If the kernel consists only of the zero vector, then the vectors are linearly independent. So, row reduction is important!

Very important are the least square problems. The **normal equation**

$$A^T A\vec{x} = A^T\vec{b}$$

has always a solution, which in general is not unique. If $\vec{x}^*$ denotes the solution, then

$$A\vec{x}^*$$

is the vector in $Img(A)$ that is closest to the vector $\vec{B}$.

This leads to the **projection onto** $Img(A)$,

$$P = A(A^T A)^{-1} A^T$$

A nicer way of computing such projections as the Gram-Schmidt procedure, which allows from a spanning set $\vec{v}_1, \ldots, \vec{v}_\ell$ to obtain an **orthonormal basis** $\vec{u}_1, \ldots, \vec{u}_k$ where $k \leq \ell$. Note that $k = \ell$ if the v-vectors form a basis.

The matrix

$$Q = [\vec{u}_1, \ldots, \vec{u}_k]$$

is an isometry and the matrix $A = [\vec{v}_1, \ldots, \vec{v}_\ell]$ can be written as

$$A = QR$$

the QR **factorization** where R is an upper triangular matrix. We have that

$$R = Q^T A \ .$$

If a subspace S is spanned by $\vec{v}_1, \ldots, \vec{v}_\ell$ then

$$QQ^T$$

is the orthogonal projection onto $Img(A)$.

Least square problems can be elegantly solved once the QR factorization is available. The equation

$$A\vec{x} = QQ^T\vec{b}$$

has always a solution, since $QQ^T\vec{b} \in Img(A)$. Hence

$$Q^T A\vec{x} = R\vec{x} = Q^T\vec{b}$$

and R is already in row reduced form.