Test I for Calculus II, Math 1502 G1-G5 , September 14, 2010

Name:

Section:

Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.
Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

Name:

Section:

Name of TA:

I: (25 points) Consider the function $f(x) = \sqrt{4+x}$.
a) (14 points) Find the 2nd order Taylor polynomial $P_2(x)$ for $f(x)$ and the corresponding remainder in Lagrange form.

$$f'(x) = \frac{1}{2}(4+x)^{-1/2} \ , \ f''(x) = -\frac{1}{4}(4+x)^{-3/2} \ , \ f'''(x) = \frac{3}{8}(4+x)^{-5/2} \ .$$

and hence

$$f(0) = 2 \ , \ f'(0) = \frac{1}{4} \ , \ f''(0) = -\frac{1}{32} \ .$$

Thus,

$$P_2(x) = 2 + \frac{1}{4}x - \frac{1}{64}x^2$$

and the remainder is given by

$$\frac{f'''(c)x^3}{3!} = \frac{1}{16(4+c)^{5/2}}$$

where c is some number between 0 and x.

b) (3 points) Using the above result compute an approximate value, call it A, for $\sqrt{5}$

$$A = P_2(1) = 2 + \frac{1}{4} - \frac{1}{64}$$

c) (8 points) Give an estimate on how accurate the value computed in b) approximates $\sqrt{5}$, i.e., give a bound on

$$|\sqrt{5} - A| \ .$$

$$|\sqrt{5} - A| \leq \frac{1}{16(4+c)^{5/2}} \leq \frac{1}{16 \times 2^5} = \frac{1}{2^9} \ .$$

Name:

Section:

Name of TA:

II: (25 points) a) (9 points) Let $f(x)$ be a continuous function on the real line. Compute

$$\lim_{x \to 0} \frac{\int_{-x}^{x} f(y)dy}{2x} \ .$$

Answer:

$$f(0)$$

Compute as well: b) (9 points)

$$\lim_{x \to 0} \frac{e^{(e^x)} - e}{x} \ ,$$

Answer:

$$\lim_{x \to 0} \frac{e^{(e^x)} - e}{x} = \lim_{x \to 0} \frac{e^{(e^x)} e^x}{1} = e \ ,$$

c) (7 points)

$$\lim_{x\to\infty} \frac{\sqrt{x^2+x+1}}{\sqrt{2x^2+1}} \ .$$

Answer:

$$\frac{1}{\sqrt{2}}$$

Name:

Section:

Name of TA:

III: (25 points) Consider the integrals

$$\int_0^\infty xe^{-x^2}\mathrm{d}x \ , \quad \int_2^\infty \frac{1}{x\log x}\mathrm{d}x$$

Write down the definition what mean by *'this integral exists'* and then decide whether they indeed exist. Compute their values if they exist. (6 points for each)

Solution:

$$\int_0^\infty xe^{-x^2}\mathrm{d}x = \lim_{A\to\infty}\int_0^A xe^{-x^2}\mathrm{d}x = \lim_{A\to\infty}\frac{1}{2}\int_0^{A^2} e^{-s}\mathrm{d}s$$
$$= \lim_{A\to\infty}\frac{1}{2}(1-e^{-A^2}) = \frac{1}{2}$$

and hence the integral exists.

$$\int_2^\infty \frac{1}{x\log x}\mathrm{d}x = \lim_{A\to\infty}\int_2^A \frac{1}{x\log x}\mathrm{d}x = \lim_{A\to\infty}\int_{\log 2}^{\log A}\frac{1}{s}\mathrm{d}s$$
$$= \lim_{A\to\infty}[\log(\log A) - \log(\log 2)]$$

which tends to $+\infty$ as $A\to\infty$. Hence the integral does not exist.

Using the comparison principle decide which of the two integrals below

exist:
b) (6 points)

$$\int_0^\infty \frac{1}{x+e^x} \mathrm{d}x$$

Solution: We expect that this integral exists since e^x grows very fast at ∞ and the denominator does not vanish anywhere. Split the integral int one from 0 to say 1 and an integral from 1 to ∞. Now

$$\int_0^1 \frac{1}{x+e^x} \mathrm{d}x$$

exists since the integrand is continuous and bounded. For the other part note that

$$\frac{1}{x+e^x} \le \frac{1}{e^x} = e^{-x}$$

and hence

$$\int_1^A \frac{1}{x+e^x} \mathrm{d}x \le \int_1^A e^{-x} \mathrm{d}x = 1 - e^{-A}$$

which converges as $A \to \infty$. Hence, by the comparison principle our integral exists.

c) (7 points)

$$\int_0^\infty \frac{1}{x+e^{-x}} \mathrm{d}x$$

This time we see that as x gets large the exponential function vanishes and we expect that the convergence of the integral is entirely determined by $1/x$, which, of course, cannot be integrated from any positive number out to infinity. Note again that there is no problem otherwise; the denominator is always strictly positive. Thus can forget about the integral from 0 to 1 On $[1, \infty)$ we know that

$$e^{-x} < 1 \leq x$$

and hence

$$\int_0^A \frac{1}{x + e^{-x}} \mathrm{d}x \geq \int_0^A \frac{1}{2x} \mathrm{d}x = \frac{1}{2} \log(A)$$

which tends to $+\infty$ as $A \to \infty$. Thus our intergal must also to $+\infty$ as A tends to ∞. Thus, our integral does not exist.

Name:

Section:

Name of TA:

IV: (25 points) Which of the following series is convergent or divergent. If it is convergent, sum it.
a) (8 points)

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+3)} .$$

Solution:

$$\sum_{k=0}^{N} \frac{1}{(k+1)(k+3)} = \frac{1}{2}\sum_{k=0}^{N}\left[\frac{1}{k+1} - \frac{1}{k+3}\right]$$

$$= \frac{1}{2}\left[\sum_{k=1}^{N+1}\frac{1}{k} - \sum_{k=3}^{N+3}\frac{1}{k}\right] = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{N+2} - \frac{1}{N+3}\right]$$

which tends to $3/4$ as $N \to \infty$. Hence the series is summable and it equals $3/4$.

b) (8 points)

$$\sum_{k=0}^{\infty} \log\frac{k+2}{k+1} .$$

Note that

$$s_N = \sum_{k=0}^{N} \log \frac{k+2}{k+1} = \sum_{k=0}^{N} [\log(k+2) - \log(k+1)] \ .$$

$= [\log 2 + \log 3 + \cdots \log(N+2)] - [\log 1 + \log 2 + \cdots + \log(N+1)] = \log(N+2)$ which diverges as $N \to \infty$.

c) (9 points) The following series converges

$$L = \sum_{k=2}^{\infty} \frac{2^k}{3^{k+1}}$$

Find L. Moreover, find the smallest n so that $0 < L - s_n < \left(\frac{2}{3}\right)^5$. Here s_n is the n-th partial sum.

Solution:

$$s_n = \sum_{k=2}^{n} \frac{2^k}{3^{k+1}} = \frac{4}{27} \sum_{k=0}^{n-2} \left(\frac{2}{3}\right)^k = \frac{4}{27} \frac{1 - \left(\frac{2}{3}\right)^{n-1}}{1 - \frac{2}{3}} = \left(\frac{2}{3}\right)^2 [1 - \left(\frac{2}{3}\right)^{n-1}]$$

As $n \to \infty$ this converges to

$$\left(\frac{2}{3}\right)^2$$

Moreover,

$$0 < \left(\frac{2}{3}\right)^2 - s_n = \left(\frac{2}{3}\right)^{n+1} < \left(\frac{2}{3}\right)^5$$

which implies that the smallest value for n is 5