

Test 2 for Calculus II, Math 1502 G1-G5 , October 5, 2010

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Section:

Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\dots$. Show your work, otherwise credit cannot be given.

Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

[illegible]

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I: Decide whether the following series converge or diverge. State which convergence test you are going to use.

a) (8 points)

$$\sum_{k=0}^{\infty} \frac{[k!]^2}{(3k)!}$$

Use the ratio test:

$$\frac{[(k+1)!]^2(3k)!}{[3(k+1)]![k!]^2} = \frac{[k!]^2(k+1)^2(3k)!}{(3k+3)(3k+2)(3k+1)(3k)![k!]^2}$$
$$\frac{(k+1)^2}{(3k+3)(3k+2)(3k+1)}$$

which tends to 0 as $k \rightarrow \infty$. Since $0 < 1$ the series converges.

b) (8 points)

$$\sum_{k=1}^{\infty} \frac{3^{k^2}}{k!}$$

Once more, use the ratio test:

$$\frac{3^{(k+1)^2}k!}{(k+1)!3^{k^2}} = \frac{3^{k^2+2k+1}k!}{k!(k+1)3^{k^2}} = \frac{3^{2k+1}}{(k+1)}$$

which tends to infinity as k tends to infinity.

c) (9 points)

$$\sum_{k=1}^{\infty} k^{-(1+\frac{1}{k})}$$

Both the root and the ratio test are inconclusive for this case. Use instead the limiting comparison test with the divergent series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

by noting that

$$\frac{k^{-(1+\frac{1}{k})}}{\frac{1}{k}} = k^{-1/k}$$

which tends to 1 as $k \rightarrow \infty$. Since $1 \neq 0$, the series in question diverges.

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II: a) (9 points) Consider the alternating series

$$L = \sum_{k=0}^{\infty} (-1)^k 10^{-k^2}$$

Find the smallest value of N so that the N -th partial sum s_N satisfies $|L - s_N| < 10^{-15}$.

Since the series is alternating we know that $|L - s_N|$ is bounded by the next term in the series after the partial sum. Hence,

$$|L - s_N| \leq 10^{-(N+1)^2} < 10^{-15}$$

$N = 3$ does the job, since $(3+1)^2 = 16$ and

$$10^{-16} < 10^{-15}.$$

b) (8 points) Find the power series expansion for $\cosh x := \frac{1}{2}(e^x + e^{-x})$.
The exponential series are

$$e^x = (1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots)$$

and

$$e^{-x} = (1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots)$$

Adding the two term by term yields

$$e^x + e^{-x} = 2(1 + \frac{x^2}{2} + \frac{x^4}{4!} + \cdots)$$

i.e., the odd powers drop out. Hence

$$\cosh(x) = (1 + \frac{x^2}{2} + \frac{x^4}{4!} + \cdots)$$

which can also be rewritten as

$$\cosh(x) = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} .$$

c) (8 points) Sum the series

$$\sum_{k=0}^{\infty} (k+2)2^{-k}$$

Note first that

$$\sum_{k=0}^{\infty} (k+2)2^{-k} = \sum_{k=1}^{\infty} k2^{-k} + 2 \sum_{k=0}^{\infty} 2^{-k}$$

The second sum is a geometric series and we know that this equals

$$2 \frac{1}{1 - \frac{1}{2}} = 4 .$$

For the first term, we always start with a related problem for which we know the answer.

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1}$$

which follows by differentiating the geometric series. Note both series converge for $|x| < 1$. Since $1/2 < 1$ we have that

$$4 = \frac{1}{(1 - 1/2)^2} = \sum_{k=1}^{\infty} k \left(\frac{1}{2}\right)^{k-1}$$

which is not quite what we want; there is a factor $1/2$ missing. Hence

$$2 = \frac{1}{2}4 = \frac{1}{2} \sum_{k=1}^{\infty} k \left(\frac{1}{2}\right)^{k-1} = \sum_{k=1}^{\infty} k \left(\frac{1}{2}\right)^k .$$

Hence the answer is 6.

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III: Find the interval of convergence of the following power series. State which convergence test you are going to use for computing the radius of convergence.

a) (8 points)

$$\sum_{k=0}^{\infty} \frac{\sqrt{k!}}{k^k} x^k$$

First we use the ratio test to test for absolute convergence. Setting

$$a_k = \frac{\sqrt{k!}}{k^k} |x|^k$$

we find

$$\frac{a_{k+1}}{a_k} = \frac{\sqrt{(k+1)!} k^k |x|}{\sqrt{k!} (k+1)^{(k+1)}} = \frac{|x|}{\sqrt{k+1}} \left(\frac{k}{k+1} \right)^k .$$

The second factor tends to e as $k \rightarrow \infty$ whereas the first factor tends to 0 as $k \rightarrow \infty$. Hence the series converges for all x .

b) (9 points)

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k} \left(\frac{x+3}{2} \right)^k$$

The same, here. For the ratio test we have to take absolute values of the terms and compute

$$\lim_{k \rightarrow \infty} \frac{k}{k+1} \left| \frac{x+3}{2} \right| = \left| \frac{x+3}{2} \right| .$$

Hence the series converges absolutely for all x with

$$\left| \frac{x+3}{2} \right| < 1 .$$

This is the open interval $(-5, -1)$. it remains to decide about the endpoints.
At $x = -1$ we get the series

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$$

which is the alternating harmonic series, which is convergent. At $x = -5$ we have the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

which diverges. Hence the interval of convergence is

$$(-5, -1] .$$

c) (8 points)

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^{-k} (x-1)^k$$

Here we use the root test and hence we have to compute

$$\lim_{k \rightarrow \infty} \left(1 + \frac{1}{k}\right) |x-1| = |x-1| .$$

Hence for all x with

$$|x-1| < 1$$

the series converges absolutely. This is the interval

$$(0, 2) .$$

At $x = 2$ we get the series

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^{-k}$$

and at $x = 0$ we get the series

$$\sum_{k=1}^{\infty} (-1)^k \left(1 + \frac{1}{k}\right)^{-k} .$$

In both cases, the series does not converge, since

$$\left(1 + \frac{1}{k}\right)^k \rightarrow e \neq 0 .$$

Hence the interval of convergence is

$$(0, 2) .$$

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IV: a) (12 points) Solve the initial value problem

$$y'' + y' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

The characteristic equation is

$$r^2 + r - 2 = 0$$

which has the roots $r_1 = -2, r_2 = 1$. The general solution is

$$y(x) = c_1 e^x + c_2 e^{-2x}.$$

Since

$$0 = y(0) = c_1 + c_2, \quad 1 = y'(0) = c_1 - 2c_2,$$

we find

$$c_1 = -c_2 = \frac{1}{3}.$$

Thus the solution is

$$y(x) = \frac{1}{3}(e^x - e^{-2x}).$$

b) (13 points) At a certain moment, a tank contains 100 liters of brine with a concentration 40 grams of salt per liter. The brine is continuously drawn off at a rate of 10 liters per minute and replaced by brine containing 20 grams salt per liter. Find the amount of salt in the tank at time t later.

Denote by $P(t)$ the amount of salt in the tank at time t . This is measured in grams. The outflow rate of brine is 10 liters per minute the inflow rate is also 10 liters per minute (that is what the word ‘replaced’ means). This means that the total amount of fluid in the tank does not change with time. The solution that flows in carries 20 grams of salt per liter.

Hence salt flows into the tank at a rate of 200 grams per minute. At time t the tank contains $P(t)$ grams of salt and hence the concentration is $\frac{P(t)}{100}$ grams per liter. The solution at that concentration flows out at a rate of 10 liters per minute.

Hence salt flows out of the tank at a rate of $\frac{P(t)}{10}$ grams per minute. Thus, we have that the rate of change of $P(t)$ equals the rate at which salt flows in minus the rate at which the salt flows out, i.e.,

$$P'(t) = 200 - \frac{P(t)}{10} ,$$

or

$$P'(t) + \frac{P(t)}{10} = 200 .$$

This differential equation can easily be solved and has the general solution

$$P(t) = 2000 + Ce^{-\frac{t}{10}} ,$$

where C is some constant. At time $t = 0$ we have 100 liters of brine containing 40 grams of salt per liter and hence

$$P(0) = 4000$$

from which we glean that $C = 2000$. Thus

$$P(t) = 2000(1 + e^{-\frac{t}{10}}).$$