Test 3 for Calculus II, Math 1502 G1-G5 , October 26, 2010

Name:

Section:

Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

Name:

Section:

Name of TA:

I: Consider the vectors $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$.

a) (6 points) Calculate $\vec{a} - \vec{b}$.

$$\vec{a} - \vec{b} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$$

b) (9 points) Calculate $|\vec{a} + \vec{b}|$.

$$\vec{a} + \vec{b} = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$$

and hence

$$|\vec{a} + \vec{b}| = \sqrt{54} = 3\sqrt{6}$$

c) (10 points) Calculate the angle between $\vec{a}$ and $\vec{b}$.

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

and hence

$$\cos\theta = \frac{10}{15} = \frac{2}{3}$$

Name:

Section:

Name of TA:

II: a) (8 points) Calculate the inverse of the matrix

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

Generally the inverse of a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is given by

$$\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The determinant is $3 \times 2 - 5 \times 1 = 1$. Hence the inverse is

$$\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

b) (8 points) Compute the matrix product $A^T A$ where

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 14 & 0 \\ 0 & 5 \end{bmatrix}$$

c) (9 points) Let $f : \mathcal{R}^2 \to \mathcal{R}^2$ be a linear transformation with

$$f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Find the matrix A_f associated with f.

$$A_f = \left[f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right]$$

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and hence

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

or

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and hence

$$A_f = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} .$$

Name:

Section:

Name of TA:

III: a) (10 points) Find the plane in $\mathcal{R}^3$, in parametrized form, that passes through the points given by the tips of the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

There are a number of ways of writing a parametrization for this plane. here is one.

$$\vec{v}_1 = \vec{e}_2 - \vec{e}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} , \quad \vec{v}_2 = \vec{e}_3 - \vec{e}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} .$$

and hence

$$\vec{x}(s,t) = \vec{e}_1 + s\vec{v}_1 + t\vec{v}_2$$

or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

As an aside, note that the equation for this plane is given by

$$x + y + z = 1$$

which provides a check.

b) (15 points) A plane is given in parametrized form by

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Find an equation for this plane. The equation has to be of the form

$$\vec{a} \cdot \vec{x} = d$$

Set

$$\vec{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

we must have

$$a + b = 0$$

and

$$b + c = 0$$

from which we glean that

$$a = -b = c$$

Thus we can choose

$$\vec{a} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Then

$$d = 1 - 2 + 3 = 2$$

and therefore the equation is

$$x - y + z = 2 \ .$$

Name:

Section:

Name of TA:

IV: Consider the system of equations

$$x - 2y + az = 2$$
$$x + y + z = 0$$
$$3y + z = 2$$

a) (15 points) For which values of a, if any, does this system have a unique solution? Find the solution for any such value of a.

The augmented matrix is

$$\begin{bmatrix} 1 & -2 & a & | & 2 \\ 1 & 1 & 1 & | & 0 \\ 0 & 3 & 1 & | & 2 \end{bmatrix}$$

First row operation leads to reduction leads to

$$\begin{bmatrix} 1 & -2 & a & | & 2 \\ 0 & 3 & 1-a & | & -2 \\ 0 & 3 & 1 & | & 2 \end{bmatrix}$$

Another one leads to

$$\begin{bmatrix} 1 & -2 & a & | & 2 \\ 0 & 3 & 1-a & | & -2 \\ 0 & 0 & a & | & 4 \end{bmatrix}$$

which is row reduced.

There is a unique solution for $a \neq 0$.

In this case it can be found by back substitution and is given by

$$z = \frac{4}{a} \ , \ y = \frac{2}{3} - \frac{4}{3a} \ , \ x = -\frac{2}{3} - \frac{8}{3a} \ .$$

b) (5 points) For which value of a, if any, does this system have infinitely many solutions?

There is no such value for a.

c) (5 points) For which value of a, if any, does this system have no solutions?

For $a = 0$.