HW 10 - PART 1

MATH 4317

FALL 2010

DR MICHAEL LOSS

CH 5, Q1 - 4

FROM INTRODUCTION TO ANALYSIS, BY MAXWELL ROSENLICHT, DOVER PUBLICATIONS, 1986

TEAM MEMBERS -

JIA LEROY & LORI KUMAR

Ch 5, Q1

b) Case 1: When $x \neq 0$

$$f'(x) = -\left(\cos\left(\frac{1}{x}\right)\right) + (2x)\sin\left(\frac{1}{x}\right)$$

Clearly the derivative of $f(x)$ exists at $x \neq 0$

Case 2: when $x = 0$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \quad x \sin \frac{1}{x}$$

The derivative exists for $x = 0$, as when $x \to 0$, it tends to 0.

Since the derivative of f exists in both cases, it is differentiable.

Ch 5 Q1 a)

Case 1 : $x \neq 0$

$$f'(x) = \sin\left(\frac{1}{x}\right) - \left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)$$

Clearly, $f'(x)$ is defined at $x \neq 0$

Case 2 : $x = 0$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \sin\left(\frac{1}{x}\right).$$

Since $x \to 0$, $f'(0)$ an infinite number of times swings between 1 and -1 ; hence $f'(x)$ is not defined at $x = 0$

Case 2 shows f is not differentiable

Ch 5 1c

$$f'(x) = \frac{1}{2|x|^{1/2}} \quad - (1)$$

As $x \to 0$, $f'(x)$ is not defined.

Hence, f is not differentiable.

Ch 5 2 a

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

$$= \frac{1}{2} \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} +$$

$$\frac{1}{2} \lim_{h \to 0} \frac{f(x_0) - f(x_0 - h)}{h}$$

$$= \frac{1}{2} f'(x_0) + \frac{1}{2} f'(x_0) = f'(x_0) \quad - (1)$$

<u>(1) is justified here</u> :

f is given to be differentiable

at x_0, so

$\forall \varepsilon > 0 \; \exists \delta > 0, \; \forall u \in U$

$0 < |u - x_0| < \delta \Rightarrow$

$$\left|\left(\frac{f(u) - f(x_0)}{(u - x_0)}\right) - f'(x_0)\right| < \varepsilon$$

Fix $\varepsilon > 0$ and let $\delta > 0$ as stated

Choose δ small enough so that

$(x_0 - \delta, x_0 + \delta) \subseteq U$, which is

possible since $x_0 \in U$ and U is open.

Taking $u = x_0 + h$, we conclude

$$0 < |h| < \delta \Rightarrow \left|\left(\frac{f(x_0 + h) - f(x_0)}{h}\right) - f'(x_0)\right| < \varepsilon$$

Taking $u = x_0 - h$, we conclude

$$0 < |h| < \delta \Rightarrow \left|\left(\frac{f(x_0 - h) - f(x_0)}{(-h)}\right) - f'(x_0)\right| < \varepsilon$$

$$\Rightarrow \left| \left(\frac{f(x_0) - f(x_0 - h)}{h} \right) - f'(x_0) \right| < \varepsilon.$$

Therefore, it is clear

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0) - f(x_0 - h)}{h} =$$

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Ch 5 2b

As $h \to 0$, $\alpha h \to 0$ and $\beta h \to 0$,

so

$$\lim_{h \to 0} \frac{f(x_0 + \alpha h) - f(x_0 + \beta h)}{h}$$

$$= \lim_{h \to 0} \frac{f(x_0 + h)}{h} - \frac{f(x_0 + h)}{h} = 0$$

Ch 5 Q 3

a) Let $v = f(x)$ and $u = f(x_0)$ -①. The step

$$\lim_{x \to x_0} \left(\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0} \right)$$

$$= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - g(x)} \cdot$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad - ⊗$$

is dubious in the proof

⊗ based on ① can be written as

$$\lim_{x \to x_0} \frac{g(v) - g(u)}{(v - u)} \cdot \frac{f(x) - f(x_0)}{(x - x_0)}$$

There is a difficulty with this step in the "proof." As $x \to x_0$, the denominator $v - u = f(x) - f(x_0)$ approaches 0, but may be equal to 0, for values of x arbitrarily close, but not equal to x_0. Whether that happens depends on the nature of the function f. The difficulty is what to do about the fact that 0 appears in the denominator for values of $v = f(x)$ other than the value that v is approaching.

(Ch 5 Q 3 b)

Here is the modified proof

$$(g \circ f)'(x_0) = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}$$

$=$

$$\Theta = \begin{cases} \dfrac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} & \text{if } f(x) \neq f(x_0) \\ g'(f(x_0)) & \text{if } f(x) = f(x_0) \end{cases}$$

$$\cdot \quad \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \Theta \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

placing the function defined above here

$$= g'(f(x_0)) \cdot f'(x_0).$$

Ch 5 Q4

Proof

Part 1

($\Rightarrow$) Let f be differentiable on an open interval in $\mathbb{R}$, (a,b), and $f'(x) \geq 0$ for $x \in (a,b)$. Let $x, y \in (a,b)$, $y > x$. Then, by the Mean Value Theorem (pg 105 Rosenlicht), there is $c \in (x,y)$ such that $f(y) - f(x) = c(y-x)$.

$f'(c) \geq 0 \Rightarrow f(y) \geq f(x)$ and so f is increasing. Similarly, if $f'(x) \leq 0$ for every $x \in (a,b)$ one gets, by reversing the inequalities, that f is decreasing.

Part 2: ($\Leftarrow$) Suppose f is differentiable and increasing on the same interval in $\mathbb{R}$, (a, b). Then $y \geq x \Rightarrow f(y) \geq f(x)$. Thus,

$$\lim_{y \to x^+} \frac{f(y) - f(x)}{y - x} \geq 0 \Rightarrow$$

$f'(x) \geq 0$ for all $x \in (a, b)$.

A similar argument shows f is decreasing $\Rightarrow f'(x) \leq 0$ for all $x \in (a, b)$.