Problem 5

Assuming the elementary properties of the trigonometric functions, show that $\tan x - x$ *is strictly increasing on* $(0, \frac{\pi}{2})$*, while the function* $\frac{\sin x}{x}$ *is strictly decreasing.*

From problem 4, we know that a differentiable real-valued function f is increasing (decreasing) where its derivative is positive (negative).

The derivative of $\tan x - x$ is $\frac{1}{\cos^2 x} - 1$. On the interval $(0, \frac{\pi}{2})$, $0 < \cos^2 x < 1$. Therefore, $\frac{1}{\cos^2 x} - 1$ is positive on this interval, so $\tan x - x$ is strictly increasing on this interval.

The derivative of $\frac{\sin x}{x}$ is $\frac{x \cos x - \sin x}{x^2}$. $x^2 > 0$ on this interval, so $\frac{x \cos x - \sin x}{x^2} < 0$ if and only if $\tan x - x > 0$. This is certainly true since at $x = 0$, this function is equal to 0, and we just saw that $\tan x - x$ is strictly increasing on the interval $(0, \frac{\pi}{2})$. Hence, $\frac{\sin x}{x}$ is decreasing on the interval $(0, \frac{\pi}{2})$.

Problem 6

Prove that a differentiable real-valued function on $\mathbb{R}$ *with bounded derivative is uniformly continuous.*

Let $f : \mathbb{R} \to \mathbb{R}$ with $|f'(x)| \leq M$, where $M \in \mathbb{R}$. By the Mean Value Theorem, there exists $c \in \mathbb{R}$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. Taking absolute values and rearranging, we see $|f(b) - f(a)| = |f'(c)||b - a| \leq M|b - a|$.

Pick $\epsilon > 0$. Set $\delta = \dfrac{\epsilon}{M}$. Then, it is clear that when $|b - a| < \delta$, $|f(b) - f(a)| \leq M|b - a| < M \cdot \dfrac{\epsilon}{M} = \epsilon$. Since there is no dependence on the point, f is uniformly continuous.

Problem 8

Let $a, b \in \mathbb{R}, a < b$*, and let* f, g *be continuous real-valued functions on* $[a, b]$ *that are differentiable on* (a, b)*. Prove that there exists a number* $c \in (a, b)$ *such that*

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$

(Hint: Consider the function

$$F(x) = (f(x) - f(a))(g(b) - g(a)) - (g(x) - g(a))(f(b) - f(a)).$$

The function $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b) since f and g are. Note that

$$F(a) = (f(a) - f(a))(g(b) - g(a)) - (g(a) - g(a))(f(b) - f(a)) = 0$$
$$F(b) = (f(b) - f(a))(g(b) - g(a)) - (g(b) - g(a))(f(b) - f(a)) = 0$$

By the Mean Value Theorem, we know that there exists $c \in (a, b)$ such that $F'(c) = \frac{F(b) - F(a)}{b - a} = 0$. Computing the derivative of $F(x)$ yields

$$F'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a))$$

So when $x = c$,

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$