Solutions to Homework 11
Chapter 5 in IA: Problems 9-14

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Problem 9

a) We use the Cauchy mean value theorm to prove the following verion of L'Hospital's rule:

Proposition. *Let U be an open interval in* **R** *and let f and g be differentiable real-valued functions on U, with g and g' nowhere zero on U. Let a be an extremity of U. Suppose that*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0, \tag{1}$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}. \tag{2}$$

Note (Cauchy): If f and g are continuous on $[a, b]$ and differentiable on $[a, b]$ and $g(a) \neq g(b)$, then

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \tag{3}$$

for some $c \in (a, b)$.

Proof.

$$\frac{f(x)}{g(x)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \tag{4}$$

for some $c \in (a, b)$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(b) - f(a)}{g(b) - g(a)} = \lim_{c \to a} \frac{f'(c)}{g'(c)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{5}$$

$\square$

b) Same as a), except that it is assumed that

$$\lim_{x \to a} \frac{1}{f(x)} = \lim_{x \to a} \frac{1}{g(x)} = 0. \tag{6}$$

Proof. If the following limit exits,

$$\lim x \to a \frac{f'(x)}{g'(x)}, \tag{7}$$

then by using Cauchy's formula gain we have that for $\epsilon > 0$ and for $a < x < z < z + \epsilon$:

$$\frac{f(x) - f(z)}{g(x) - g(z)} = \frac{f(x)}{g(x} \left[\frac{1 - \frac{f(z)}{f(x)}}{1 - \frac{g(z)}{g(x)}} \right] = \frac{f'(c)}{g'(c)} \tag{8}$$

for some $c \in (a, a + \epsilon)$.

Now taking z close to a and fixed and if the limit (Equation 7) exists, then we know that,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{c \to a} \frac{f'(c)}{g'(c)} \left[\frac{1 - \frac{g(z)}{g(x)}}{1 - \frac{f(z)}{f(x)}} \right] \tag{9}$$

for $x < c < z$ implies

$$\lim_{x \to a} \left[\frac{1 - \frac{g(z)}{g(x)}}{1 - \frac{f(z)}{f(x)}} \right] = 1, \tag{10}$$

so that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{c \to a} \frac{f'(c)}{g'(c)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}. \tag{11}$$

$\square$

c) Same as a), except $U = \{x \in \mathbf{R} : x > \alpha\}$ for some $\alpha \in \mathbf{R}$ and

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0. \tag{12}$$

Proof. The proof is similar to part a), using the theorem from Problem 8 on page 91, namely

$$\lim_{x \to \infty} f(x) = \lim_{y \to 0} f_1(y), \tag{13}$$

where $f_1 : (0, \frac{1}{\alpha} \to \mathbf{R}$ is given by $f_1(y) = f(\frac{1}{y})$ if the latter limit exist, our problem then becomes:

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f_1(y)}{g_1(y)}, \tag{14}$$

which is a special case of L'Hospital's rule from part a) with $a = 0$. Therefore,

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f_1(y)}{g_1(y)} = \lim_{y \to 0} \frac{f_1(y) - f_1(0)}{g_1(y) - g_1(0)} = \lim_{c \to 0} \frac{f_1'(c)}{g_1'(c)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} \tag{15}$$

for $0 < c < y$. $\square$

d) Same as c), expect that it is assumed that

$$\lim_{x \to a} \frac{1}{f(x)} = \lim_{x \to a} \frac{1}{g(x)} = 0 \quad (\text{or } \lim_{x \to \infty} f(x) = \infty). \tag{16}$$

Proof. We have that

$$\lim_{x \to \infty} \frac{\frac{1}{f(x)}}{\frac{1}{g(x)}} = \lim_{x \to \infty} \frac{g(x)}{f(x)}. \tag{17}$$

Again proceeding as in part c) make the change of variable $x = \frac{1}{y}$ so that

$$\lim_{x \to \infty} f(x) = \lim_{y \to 0} f(\frac{1}{y}), \tag{18}$$

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such that,

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{y\to 0}\frac{f(\frac{1}{y})}{g(\frac{1}{y})} = \lim_{y\to 0} = \lim_{y\to 0}\frac{-\frac{1}{y^2}f'(\frac{1}{y})}{-\frac{1}{y^2}g'(\frac{1}{y})} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}, \tag{19}$$

since $\frac{\alpha x}{\alpha y} = -\frac{1}{y^2}$.

Therefore,

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}. \tag{20}$$

$\square$

Problem 10

Proposition. *Let U and V be open subsets of $\mathbf{R}$ and let $g : U \to V$ and $f : V \to \mathbf{R}$ be functions. Let $x_0 \in U$ such that $g^{(n)}(x_0)$ and $f^{(n)}(g(x_0))$ exist. Then $(f \circ g)^{(n)}(x_0)$ exists.*

Proof. Let $f^{(n)}(x)$ be the nth derivative of f and similarly for g.

Note: If both f and g are continuous and differentiable, then by the proposition on page 103 $(f \circ g)(x)$ is differentiable and $(f \circ g)'(x) = f'(g(x))g'(x)$.

This will be proved by induction on $(f \circ g)(x_0)$.

Fact: $f'(g(x_0))$ and $g'(x_0)$ exist by the assumption in the problem statement. Then for $n = 1$,

$$(f \circ g)'(x_0) = \lim_{x\to x_0}\frac{f(g(x)) - f(g(x_0))}{x - x_0} = f'(g(x_0))g'(x_0), \tag{21}$$

and for $n = 2$,

$$\begin{aligned}(f \circ g)^{(2)}(x_0) &= \frac{d}{dx}\left[f'(g(x_0))g'(x_0)\right] = f'(g(x_0))g''(x_0) + g'(x_0)f''(g(x_0))g'(x_0) \\ &= f'(g(x_0))g''(x_0) + f''(g(x_0))[g'(x_0)]^2.\end{aligned} \tag{22}$$

By assumption $f'(g(x_0)), f''(g(x_0)), g'(x_0)$, and $g''(x_0)$ along with their sums and products are all differentiable. (By the propositon on page 101 the sum or product of two differntiable functions is itself differentiable.) Continuing on in this fashion, the nth derivative will be some combination of sums and products of derivatives of f and g of orders 1 through n.

For those interested, a closed form expresssion for $\frac{d^n}{dx^n}(f \circ g)(x_0)$ is given by Faá di Bruno's identity:

$$\frac{d^n}{dx^n}(f \circ g)(x_0) = \sum \frac{n!}{m_1!m_2!m_3!\dots}f^{(m_1+m_2+\dots+m_n)}(g(x_0)) \cdot \prod_{j=1}^{n}\left(\frac{g^{(j)}(x_0)}{j!}\right)^{m_j}, \tag{23}$$

where the sum is over all n-tuples of nonnegative integers $(m_1,\dots,m_n)$, where $1\cdot m_1 + 2\cdot m_2 + \dots + n\cdot m_n = n$.

$\square$

Problem 11

Proposition. *Let $f : U \to \mathbf{R}, U \in \mathbf{R}$ twice differentiable at $x_0 \in U$. If $f'(x_0) = 0$ and $f''(x_0) < 0$ $(f''(x_0) > 0)$ then the restriction of f to $B(x_o,\delta)$ attains a maximum (minimum) at x_0.*

Proof. We know from the definition of differentiability at x_0 and applying $f'(x_0) = 0$ and $f''(x_0) < 0$ that

$$\lim_{x\to x_0}\frac{f'(x) - f'(x_0)}{x - x_0} = f''(x_0)$$

$$\lim_{x\to x_0}\frac{f'(x)}{x - x_0} < 0. \tag{24}$$

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If $x_0 < x$ then $(x - x_0) > 0$ and $f'(x) < 0$ satisfies the inequality. Therefore, f is decreasing for $x > x_0$. Likewise, if $x_0 > x$ then $(x - x_0) < 0$ and $f'(x) > 0$ satisfies the inequality. Therefore, f is increasing for $x < x_0$. Since f is increasing as we approach x_0 from the left and decreasing as we move away from x_0 towards the right, $f(x_0)$ must be a maximum.

Note: Since f is differentiable at x_0, it is also continous at x_0, meaning that if we restrict f to $B(x_0, \delta)$, then we know that there are no discontinuities for f restricted to that ball and our analysis holds.

It is also true that if $f''(x_0) > 0$ that $f(x_0)$ is a minimum. To satisfy the inequality, $f'(x)$ has to have the same sign as $x - x_0$, meaning that as x approaches x_0 from the left, f decreases and as x moves away from x_0 to the right, f is increasing, i.e. $f(x_0)$ must be a minimum. $\square$

Problem 12

This problem can be split into the following three parts:

a) First,

Proposition. $f : U \to \mathbf{R}, U \in \mathbf{R}$ *is called convex if no point on the line segement between any two points of its graph lies below the graph.*

Proof. We first need an inequality for testing whether a point on the line between any two points of its graph lies below the graph,

$$f(\gamma x + (1 - \gamma)y) \le \gamma f(x) + (1 - \gamma)f(y). \tag{25}$$

Let $\gamma \in [0, 1]$ and without loss of generality assume $x < y$. If $\gamma = 0$, then $f(y) \le f(y)$ and if $\gamma = 1$, then $f(x) \le f(x)$; therefore, our equation holds at the boundaries. Let $\gamma = \frac{1}{2}$ (midpoint convex), then

$$f\left(\frac{1}{2}x + (1 - \frac{1}{2})y\right) \le \frac{1}{2}f(x) + (1 - \frac{1}{2})f(y).$$
$$f\left(\frac{x + y}{2}\right) \le \frac{f(x) + f(y)}{2}. \tag{26}$$

It is a fact that the limit of a convex function is

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}, \tag{27}$$

and this limit exists. We can the show that if f is convex it is also continuous on U. For any $\epsilon > 0$, there exits $\delta > 0$ with $|x - x_0| < \delta$, such that $|f(x) - f(x_0)| < \epsilon$. We can rewrite $|f(x) - f(x_0)|$ as $|f(x + h) - f(x)|$, such that we have,

$$\lim_{h \to 0} f(x + h) - f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \lim_{h \to 0} h = 0, \tag{28}$$

meaning that when x_0 and x are close, so are $f(x_0)$ and $f(x)$, which is by definition continuous.

It is known that if f is midpoint convex and continous (see "3. Convex Functions" in *Distributions and Fourier Transforms* by Donoghue), then f is convex. We know this to be true, since if f were not midpoint convex, Equation 25 would be invalid for $\gamma = \frac{1}{2}$, which would contradict our assumption that f is convex. Likewise, if f were midpoint convex, but not continous, a disontinuity of f on the interval could contradict the inequality in Equation 25, which again would contradict our assumption that f is a convex function on the open inverval U in $\mathbf{R}$.

$\square$

b) Second,

Proposition. *If $f : U \to \mathbf{R}, U \in \mathbf{R}$ is differentiable, then f is convex if no point of the graph lie below any point of any tangent to the graph.*

Proof. We showed in part a) that if f is convex, then $f(\gamma x + (1 - \gamma)y) \leq \gamma f(x) + (1 - \gamma)f(y)$ holds. Let's assume this statement also holds for a differentiable function f. The inequality that we need to satisfy is,

$$f(y) \geq f(x) + f'(x)(y - x). \tag{29}$$

Then, by replacing $y = x + h$ we get,

$$
\begin{aligned}
f(\gamma x + (1 - \gamma)y) &\leq \gamma f(x) + (1 - \gamma)f(y) \\
f(\gamma x + (1 - \gamma)(x + h)) &\leq \gamma f(x) + (1 - \gamma)f(x + h) \\
f(x + (1 - \gamma)h) - f(x) &\leq (1 - \gamma)(f(x + h) - f(x)) \\
\frac{f(x + (1 - \gamma)h) - f(x)}{(1 - \gamma)} &\leq f(x + h) - f(x)
\end{aligned}
\tag{30}
$$

We can then take the limit as $\gamma \to 1$,

$$
\begin{aligned}
\lim_{\gamma \to 1} h \frac{f(x + (1 - \gamma)h) - f(x)}{(1 - \gamma)h} &\leq f(x + h) - f(x) \\
hf'(x) &\leq f(x + h) - f(x) \\
f(x + h) &\geq f(x) + hf'(x) \\
f(y) &\geq f(x) + f'(x)(y - x)
\end{aligned}
\tag{31}
$$

which is the inequality we need. $\qquad \square$

c) Last,

Proposition. *If $f : U \to \mathbf{R}, U \in \mathbf{R}$ is twice differentiable, then f is convex if $f''(x)$ is nonnegative at all points.*

Proof. We showed in part b) that if f is convex, then $f(y) \geq f(x) + f'(x)(y - x)$ holds. Let's assume this statement also holds for a twice differentiable function f. Then,

$$
\begin{aligned}
f(x) - f(y) &\geq f'(y)(x - y) \\
-(f(y) - f(x)) &\geq -(y - x)f'(y) \\
f(y) - f(x) &\leq f'(y)(y - x).
\end{aligned}
\tag{32}
$$

Therefore, we have that

$$
\begin{aligned}
f'(x)(y - x) \leq f(x) - f(y) &\leq f'(y)(y - x) \\
f'(x)(y - x) &\leq f'(y)(y - x) \\
f'(x) &\leq f'(y).
\end{aligned}
\tag{33}
$$

Since the first derivative of f is increasing, $f''(x) \geq 0$, which means that f is convex.

$\qquad \square$

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Problem 13

Proposition. *Let $f : U \to \mathbf{R}$ be n-times differentiable at the point $x_0 \in U$, then*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - f'(x_0)\frac{h}{1!} - \ldots - f^{(n-1)}(x_0)\frac{h^{n-1}}{(n-1)!}}{h^n} = \frac{f^{(n)}(x_0)}{n!}. \tag{34}$$

Proof. Differentiating numerator and denominator,

$$\left(\because \lim_{h \to 0} h^n = \lim_{h \to 0} f(x_0 + n) - f(x_0) - f'(x_0)\frac{h}{1!} - \ldots - f^{(n-1)}(x_0)\frac{h^{n-1}}{(n-1)!} = f(x_0 + 0) - f(x_0) = 0 \right) \tag{35}$$

$$\lim_{h \to 0} \frac{f'(x_0 + h) - f'(x_0)\frac{h}{1!} - \ldots - f^{(n-1)}(x_0)\frac{h^{n-1}}{(n-2)!}}{nh^{n-1}}. \tag{36}$$

Differentiating with respect to h n times,

$$\lim_{h \to 0} \frac{f^{(n)}(x_0 + h)}{n!} = \frac{f^{(n)}(x_0)}{n!}. \tag{37}$$

$\square$

Problem 14

Proof. Let $f : \mathbf{R} \to \mathbf{R}$ be a function such that $f(y) = (a + y)^n$. We know $f^{(k)} =$ for $k > n$ and can apply Taylor's theorem,

$$f(\beta) = f(\alpha) + \frac{f'(\alpha)}{1!}(\beta - \alpha) + \frac{f''(\alpha)}{2!}(\beta - \alpha)^2 + \ldots + \frac{f^{(n+1)}}{(n+1)!}(c)(\beta - \alpha)^{n+1} \tag{38}$$

with $\beta = x$ and $\alpha = 0$ (as $0, x \in \mathbf{R}$). Therefore,

$$f(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \ldots + x^n. \tag{39}$$

$\square$