MATH 4317 Fall 2010
HOMEWORK # 12

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Chapter 6: 12-17, 19, 21, 22

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12 **Claim.** If $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then $\int_a^b f(x)\,dx = f'(\xi)(b-a)$ for some $\xi \in [a,b]$.

Proof. f is continuous on $(a,b) \subset [a,b]$ so there exists a $F\colon (a,b) \to \mathbb{R}$ such that $F' = f$ and $\int_a^b f(x)\,dx = F(b) - F(a)$. F is differentiable (and therefore continuous) on (a,b), so by the Mean Value Theorem, there exists a $\xi \in (a,b)$ such that,

$$\begin{aligned} F'(\xi) &= \frac{F(b)-F(a)}{b-a} \\ f(\xi) &= \frac{\int_a^b f(x)\,dx}{b-a} \\ \int_a^b f(x)\,dx &= f(\xi)(b-a) \end{aligned}$$

□

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13 **Claim.** If $f\colon [a,b] \to \mathbb{R}$ is continuous, and $f(x) \geq 0$, then

$$\lim_{n\to\infty} \left(\int_a^b (f(x))^n\,dx \right)^{\frac{1}{n}} = \max\{f(x) : x \in [a,b]\}$$

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Proof. Let $M = \max\{f(x) : x \in [a,b]\}$. Then $f(x) \leq M$ implies $(f(x))^n \leq M^n$. Because $|(f(x))^n| = (f(x))^n$, it follows from the previous inequality that $\left|\int_a^b (f(x))^n\,dx\right| \leq M^n|b-a|$. Since $f(x) \geq 0$,

$$\int_a^b (f(x))^n\,dx = \left| \int_a^b (f(x))^n\,dx \right| \leq M^n|b-a|$$

Therefore, $\left(\int_a^b (f(x))^n\,dx\right)^{\frac{1}{n}} \leq M|b-a|^{\frac{1}{n}}$. Now, consider:

$$\begin{aligned} \lim_{n\to\infty} \left(\int_a^b (f(x))^n\,dx \right)^{\frac{1}{n}} &\leq \lim_{n\to\infty} M|b-a|^{\frac{1}{n}} \\ \lim_{n\to\infty} \left(\int_a^b (f(x))^n\,dx \right)^{\frac{1}{n}} &\leq M \end{aligned}$$

□

14 **Claim.** If f is a continuous real-valued function on $\{x \in \mathbb{R} : x \geq 0\}$ and $\lim_{x\to+\infty} f(x) = c$, then:

$$\lim_{x\to+\infty} \frac{1}{x}\int_0^x f(t)\ dt = c.$$

Proof. Since $\lim_{x\to+\infty} f(x) = c$, $|f(n) - c| < \epsilon$ for some N with $n > N$,

$$\begin{aligned}
\lim_{x\to+\infty} \frac{1}{x}\int_0^x f(t)dt &= \lim_{x\to+\infty}\left[\frac{1}{x}\int_0^n f(t)\ dt + \frac{1}{x}\int_0^x c\ dt\right] \\
&= \lim_{x\to+\infty}\left[\frac{1}{x}\left(F(n) - F(0)\right)x + \frac{1}{x}(cx - cn)\right] \\
&= \lim_{x\to+\infty}\left[\frac{F(n) - F(0) - cn}{x} + c\right] = c
\end{aligned}$$

Thus,

$$\lim_{x\to+\infty} \frac{1}{x}\int_0^x f(t)dt = c$$

□

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15 **Claim.** Let $[a, b]$ and $[c, d]$ be closed intervals in $\mathbb{R}$, and let f be a continuous real-valued function on $\{(x, y) \in E^2 : x \in [a, b], y \in [c, d]\}$. Show that the function $g\colon [c, d] \to \mathbb{R}$ defined by $g(y) = \int_a^b f(x, y)dx$ for all $y \in [c, d]$ is continuous.

Proof. Since $\{(x, y) \in E^2 : x \in [a, b], y \in [c, d]\}$ is compact, f is uniformly continuous. Thus, for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x, y) - f(x, y_0)| < \epsilon$ for all $x \in [a, b]$ whenever $|y - y_0| < \delta$ with $y, y_0 \in [c, d]$.

If f is integrable on $[a, b]$ then so is $|f|$. Furthermore, $|\int_a^b f(x)dx| \leq \int_a^b |f(x)dx|$.

Thus,

$$|g(y) - g(y_0)| = \left|\int_a^b f(x, y)dx - f(x, y_0)dx\right| \leq \int_a^b |f(x, y)dx - f(x, y_0)dx| < \epsilon(b - a)$$

whenever $|y - y_0| < \delta$. Hence, the restriction of g to $[c, d]$ is continuous. □

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16 **Claim.** The real-valued function F on $C[a, b]$ which sends any function f into $\int_a^b f(x)dx$ is uniformly continuous.

Proof. For all $\epsilon > 0$, there exists a positive $\delta = \frac{\epsilon}{|b-a|}$ such that if $f, g \in C[a,b]$ and $d(f,g) < \delta$, then

$$\begin{aligned} d(F(f), F(g)) = |F(f) - F(g)| &= \left| \int_a^b f(x)\ dx - \int_a^b g(x)\ dx \right| = \left| \int_a^b f(x) - g(x)\ dx \right| \\ &\leq \max_{x \in [a,b]} \{|f(x) - g(x)|\} |b-a| = d(f,g)|b-a| < \delta|b-a| = \epsilon \end{aligned}$$

Since $d(F(f), F(g)) < \epsilon$ for any $f, g \in C[a,b]$, F is uniformly continuous. $\square$

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17 **Claim.** If u and v are real-valued functions on an open subset of $\mathbb{R}$ containing the interval $[a,b]$ and if u and v have continuous derivatives, then

$$\int_a^b u(x)v'(x)dx = u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)\ dx.$$

Proof. By the product rule, $\frac{d}{dx} u\left(x\right) v\left(x\right) = u(x)v'(x) + u'(x)v(x)$. Integrate over $[a,b]$,

$$\begin{aligned} u(b)v(b) - u(a)v(a) &= \int_a^b \left(\frac{d}{dx} u(x)v(x) \right) dx \\ &= \int_a^b \left[u(x)v'(x) + u'(x)v(x) \right] dx \\ &= \int_a^b \left[u(x)v'(x) \right] dx + \int_a^b \left[u'(x)v(x) \right] dx \end{aligned}$$

Rearranging gives

$$\int_a^b u(x)v'(x)dx = u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)\ dx$$

$\square$

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19 **Claim.** If f is a function on the open interval $U \subset \mathbb{R}$ to $\mathbb{R}$ with a continuous $(n+1)$ derivative on U, then for any $a, b \in U$, we have:

$$\begin{aligned} f(b) = f(a) + \frac{f'(a)(b-a)}{1!} + \frac{f''(a)(b-a)^2}{2!} + \cdots \\ + \frac{f^{(n)}(a)(b-a)^n}{n!} + \frac{1}{n!} \int_a^b f^{(n+1)}(x)(b-x)^n\ dx \end{aligned}$$

Proof. For any $a, b \in U$, define $R_n(b,a) \in \mathbb{R}$ by

$$f(b) = f(a) + \frac{f'(a)(b-a)}{1!} + \frac{f''(a)(b-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(b-a)^n}{n!} + R_n(b,a)$$

It will suffice to show that $R_n(b,a) = \frac{1}{n!}\int_a^b f^{(n+1)}(x)(b-x)^n\,dx$. By the lemma in §5.4,

$$\frac{d}{dx}R_n(b,x) = -\frac{f^{(n+1)}(x)(b-x)^n}{n!}$$

Integrating gives:

$$\begin{aligned}
\int_a^b \frac{d}{dx}R_n(b,x)\,dx &= -\frac{1}{n!}\int_a^b f^{(n+1)}(x)(b-x)^n\,dx \\
R_n(b,b) - R_n(b,a) &= -\frac{1}{n!}\int_a^b f^{(n+1)}(x)(b-x)^n\,dx
\end{aligned}$$

However, $R_n(b,b) = f(b) - f(b) = 0$ from the definition, so

$$R_n(b,a) = \frac{1}{n!}\int_a^b f^{(n+1)}(x)(b-x)^n\,dx \qquad \square$$

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21 (a) First, rewrite the expression in a more familiar form:

$$\begin{aligned}
\lim_{n\to\infty}\frac{1^k+2^k+\cdots+n^k}{n^xk+1} &= \lim_{n\to\infty}\left(\left(\frac{1}{n}\right)^k+\left(\frac{2}{n}\right)^k+\cdots+\left(\frac{n}{n}\right)^k\right)\frac{1}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^k\frac{1}{n}
\end{aligned}$$

This is the limit of the Riemann sum corresponding to $f(x) = x^k$, $k \in \mathbb{R}, k > 0$ and partition $P = (0, \frac{1}{n}, \frac{2}{n}, \ldots, 1)$. Therefore,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^k\frac{1}{n} = \int_0^1 x^k\,dx = \left[\frac{x^k}{k+1}\right]_0^1 = \frac{1}{k+1}$$

(b) Again, rewrite in a clearer form,

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right) &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+i/n} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+i/n}\frac{1}{n}
\end{aligned}$$

Like above, this is the limit of a Riemann sum over the partition $P = (0, \frac{1}{n}, \frac{2}{n}, \ldots, 1)$. In this case, $f(x) = \frac{1}{1+x}$. Thus,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+i/n}\frac{1}{n} = \int_0^1\frac{1}{1+x}\,dx = [\ln(1+x)]_0^1 = \ln 2$$

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22 **Claim.** For $n = 1, 2, 3, \ldots$, the number

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$$

is positive, decreases, and hence the sequence of these numbers converge to a limit between 0 and 1.

Proof. First, consider the left hand Riemann sum of $\frac{1}{x}$ over $[1, n]$. Because $\frac{1}{x}$ is monotonically decreasing for $x > 0$, the left hand Riemann sum will overestimate $\int_1^n \frac{1}{x}\, dx$. In particular, if the partition has width 1, then,

$$\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{1}{i} > \int_1^n \frac{1}{x}\, dx = \log n$$

$$\sum_{i=1}^{n} \frac{1}{i} > \log n$$

$$\sum_{i=1}^{n} \frac{1}{i} - \log n > 0$$

We would like to show that:

$$\sum_{i=1}^{n+1} \frac{1}{i} - \log(n+1) < \sum_{i=1}^{n} \frac{1}{i} - \log(n).$$

$\log\left(\left(1 + \frac{1}{n}\right)^{n+1}\right)$ is strictly decreasing since $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e < \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n+1}$.

$$1 = \log(e) < \log\left(\left(1 + \frac{1}{n}\right)^{n+1}\right)$$

$$\frac{n+1}{n+1} < (n+1)\log\left(\frac{n+1}{n}\right)$$

$$\frac{1}{n+1} < \log\left(\frac{n+1}{n}\right)$$

$$\frac{1}{n+1} < \log(n+1) - \log(n)$$

$$\frac{1}{n+1} - \log(n+1) < -\log(n)$$

Thus, as desired:

$$\sum_{i=1}^{n+1} \frac{1}{i} - \log(n+1) < \sum_{i=1}^{n} \frac{1}{i} - \log(n).$$

Since the sequence is decreasing with initial value $1 - \log 1 = 1$ and lower bound 0, we know that $f(n) - g(n)$ converges to some number in $[0, 1]$, Euler's constant. $\square$