Week # 14 Solutions Tucker Moore, Belal Al-Khalil, George Mason

20 First, prove that, under the given conditions

$$\int_a^b \sqrt{\left(f_1'(x)\right)^2 + \cdots + \left(f_n'(x)\right)^2}\, dx$$

exists.

Since each of $f_1', \dots, f_n'$ are continuous and square and square-root are continuous the integrand is continuous. Therefore, the integral exists.

Now, show that the integral is equal to

$$\text{l.u.b.} \left\{ \sum_{i=1}^{N} d(f(x_{i-1}), f(x_i)) : x_0, \dots, x_N \text{ a partition of } [a,b] \right\}.$$

Pick $\epsilon > 0$.

Since the integral exists, there exists a number $\delta > 0$ such that

$$\left| \int_a^b \sqrt{\left(f_1'(x)\right)^2 + \cdots + \left(f_n'(x)\right)^2}\, dx - \sum_{\substack{i=1 \\ x_{i-1} \le x_i^* \le x_i}}^{N} \sqrt{\left(f_1'(x_i^*)\right)^2 + \cdots + \left(f_n'(x_i^*)\right)^2}\,(x_i - x_{i-1}) \right| < \epsilon$$

whenever $a = x_0 < x_1 < \cdots < x_N = b$ is a partition with width less than δ.

Manipulation of the Riemann Sum brings the quantity $(x_i - x_{i-1})$ inside.

$$\sum_{i=1}^{N} \sqrt{\sum_{j=1}^{n} \left(f_j'(x_i^*)(x_i - x_{i-1}) \right)^2}$$

The mean value theorem provides for the existance of numbers

$x_{i,j}^*$ $i = 1, 2, \dots, N$ $j = 1, 2, \dots, n$ $x_{i-1} \le x_{i_j}^* \le x_i$

such that

$$f_j'(x_{i_j}^*)(x_i - x_{i-1}) = f_j(x_i) - f_j(x_{i-1}) \quad \forall\ i = 1, \dots, N,\ j = 1, \dots, n.$$

Pick $\eta > 0$.

Since $f_1', \dots, f_n'$ are continuous, there exist $\delta_1, \delta_2, \dots, \delta_n$ such that

$$\left| f_j'(x_{i_j}^*) - f_j'(x_i^*) \right| < \eta \quad \text{when} \quad \left| x_{i_j}^* - x_i^* \right| < \delta_j$$

Fix $\delta < \min\{1, \delta_1, \delta_2, \dots, \delta_n\}$

$$\left| f_j'(x_{ij}^*)(x_i - x_{i-1}) - f_j'(x_i^*)(x_i - x_{i-1}) \right|$$
$$= \left| f_j'(x_{ij}^*) - f_j'(x_i^*) \right| \left| x_i - x_{i-1} \right|$$
$$< \eta \delta$$
$$< \eta \quad (\text{since } \delta < 1)$$

This means that

$$\left| f_j'(x_i^*)(x_i - x_{i-1}) - \left(f_j(x_i) - f_j(x_{i-1}) \right) \right| < \eta$$

when the partition has width less than δ. Since η is arbitrary we have

$$\sum_{i=1}^{N} \sqrt{\sum_{j=1}^{n} \left(f_j'(x_i^*) \right)^2}\, (x_i - x_{i-1}) = \sum_{i=1}^{N} \sqrt{\sum_{j=1}^{n} \left(f_j(x_i) - f_j(x_{i-1}) \right)^2}$$

as the width of the partition goes to zero.

$$\sqrt{\sum_{j=1}^{n} \left(f_j(x_i) - f_j(x_{i-1}) \right)^2} = d\left(f(x_i), f(x_{i-1}) \right) \quad \text{so}$$

$$\int_a^b \sqrt{(f_1'(x))^2 + \cdots + (f_n'(x))^2}\, dx = \lim_{\delta \to 0} \sum_{i=1}^{N} d(f(x_i), f(x_{i-1}))$$

The triangle inequality gives that $\sum_{i=1}^{N} d(f(x_i), f(x_{i-1}))$ increases as the width of the partition decreases, so the limit is the l.u.b.

$$\int_a^b \sqrt{(f_1'(x))^2 + \cdots + (f_n'(x))^2}\, dx = \text{l.u.b.} \left\{ \sum_{i=1}^{N} d(f(x_i), f(x_{i-1})) : \{x_i\} \text{ a partition} \right\}$$

23 $\quad f' = f \qquad f(0) = 1$

Let $g(x) = e^{-x} f(x)$

$f(x) = e^{x} g(x)$

$e^{x}g(x) = f(x) = f'(x) = e^{x} g(x) + e^{x} g'(x)$

This implies

$e^{x} g'(x) = 0$

$g'(x) = 0$

$g(x) \equiv A \qquad$ (constant)

so $\; f(x) = A e^{x}$

$f(0) = 1 \Rightarrow A = 1$

so $\; f(x) = e^{x}$

24 a) $\log(1+x) \leq x \qquad x > -1, \quad \log(1+x) = x$ iff $x = 0$

equivalent to

$x - \log(1+x) \geq 0 \qquad x > -1$

Find the minimum of $x - \log(1+x)$, $x > -1$

The minimum does not occur at the endpoints ($x=-1$, $x=+\infty$) because the value becomes arbitrarily large there. So, differentiate

$$1 - \frac{1}{1+x} = 0 \Rightarrow 1 + x = 1 \Rightarrow x = 0.$$

So $\; x - \log(1+x) \geq 0 - \log(1+0) = 0$

b) $e^{x} \geq 1 + x \quad \forall x \qquad e^{x} = 1 + x$ iff $x = 0$

When $x > -1$, this holds due to part (a).

When $x \leq -1$, $1 + x \leq 0$ and $e^{x} > 0$ so $e^{x} > 1 + x$, $x \leq -1$

25 a) Prove $\lim_{x\to\infty} \frac{\log x}{x^\alpha} = 0$ if $\alpha > 0$

Look at $x > 1$

$$\frac{1}{x} < \frac{1}{x^{1-\beta}} \quad , \beta > 0$$

$$\log x = \int_1^x \frac{1}{t}\,dt < \int_1^x t^{-1+\beta}\,dt = \frac{1}{\beta}\left(x^\beta - 1\right)$$

Let $\beta = \frac{\alpha}{2}$

$$\frac{\log x}{x^\alpha} \le \frac{1}{x^\alpha} \quad \frac{2}{\alpha}\left(x^{\frac{\alpha}{2}} - 1\right) = \frac{2}{\alpha}\left(x^{-\frac{\alpha}{2}} - x^{-\alpha}\right) \xrightarrow{x\to+\infty} 0$$

$$0 \le \lim_{x\to+\infty} \frac{\log x}{x^\alpha} \le 0$$

This also shows $\frac{\log x}{x} \to 0$, let $\alpha = 1$.

b) $\lim_{x\to 0} x^\alpha \log x = 0$ if $\alpha > 0$

$$\lim_{x\to 0} x^\alpha \log x = \lim_{x\to+\infty} \frac{-\log x}{x^\alpha} = -\lim_{x\to+\infty} \frac{\log x}{x^\alpha} = 0$$

c) $\lim_{x\to+\infty} \frac{x^n}{e^x} = 0 \quad \forall n \in \mathbb{R}$

We know $\lim_{x\to\infty} \frac{\log x}{x^\alpha} = 0$

let $y = \log x \quad (x = e^y)$

$$= \lim_{y\to\infty} \frac{y}{e^{\alpha y}} = 0$$

Let $\alpha = \frac{1}{n}$

$$= \lim_{y\to\infty} \frac{y}{e^{\frac{1}{n}y}} = 0$$

So $\lim_{y\to\infty} \frac{y^n}{e^y} = \lim_{y\to\infty} \left(\frac{y}{e^{\frac{1}{n}y}}\right)^n = 0$

26 $\quad f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x > 0 \\ 0 & x \le 0 \end{cases}$

Differentiability is obvious except at $x=0$. We differentiate each branch and then check if they agree at $x=0$. (This works since $\lim_{x\to 0} e^{-\frac{1}{x^2}} = 0$)

$$f'(x) = \begin{cases} \frac{2}{x^3} e^{-\frac{1}{x^2}} & x > 0 \\ 0 & x \le 0 \end{cases}$$

$\lim_{x \to 0} \frac{2}{x^3} e^{-\frac{1}{x^2}} = 0 \quad$ (Consequence of 25(c))

$f^{(1)}(x)$ exists and is continuous $\quad f^{(1)}(0) = 0$

[illegible] exists and is continuous.

$$f^{(n)}(x) = \begin{cases} P_{3n}\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} & x > 0 \\ 0 & x \le 0 \end{cases}$$

$P_k(t)$ is a k^{th} order polynomial of t.

$\lim_{x \to 0} P_{3n}\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} = 0 \ \forall n \in \{0,1,\dots\}$ as a consequence of 25(c).

Assume $f^{(n-1)}(x) = \begin{cases} P_{3(n-1)}\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} & x > 0 \\ 0 & x \le 0 \end{cases}$

$$f^{(n+1)}(x) = \frac{d}{dx} P_{3n}\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} = P_{3n+1}\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} + \frac{2}{x^3} P_{3n}\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$$

$$= P_{3n+1}\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} + P_{3n+3}\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$$

$$= P_{3(n+1)}\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$$

So, by induction $f^{(n)}(x)$ exists, cont and $f^{(n)}(0) = 0 \quad \forall n \in \{0,1,2,\dots\}$

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$g(x) \geq 0$ (otherwise $|f(x)| \leq g(x)$ would not be possible.

$$\int_a^\infty g(x)\,dx = \lim_{y\to\infty} \int_a^y g(x)\,dx \quad \text{exists}$$

Pf: Pick $\varepsilon > 0$ Then there exists L such that

$$\int_{L_1}^{L_2} g(x)\,dx < \varepsilon \text{ for all } L_2 > L_1 > L.$$

Let $L_1 \to L$, $L_2 \to \infty$, look at $|f(x)|$

$$\int_L^\infty |f(x)|\,dx < \int_L^\infty g(x)\,dx < \varepsilon$$

Now look at:

$$\int_a^{L'} f(x)\,dx = \int_a^L f(x)\,dx + \int_L^{L'} f(x)\,dx \qquad L' > L$$

$$\left| \int_L^{L'} f(x)\,dx \right| \leq \int_L^{L'} |f(x)|\,dx < \varepsilon$$

Let $L' \to \infty$

$$\int_a^\infty f(x)\,dx < \int_a^L f(x)\,dx + \varepsilon$$

$$|x^\alpha f(x)| \leq M \Rightarrow |f(x)| \leq \frac{M}{x^\alpha} \qquad \alpha > 1$$

$$|f(x)| \xrightarrow{x\to\infty} 0$$