Math 4317
Homework #2
Solutions

Created by:

Sarwat Nasir
Christopher Stevens
Eric Tollefson

Submitted: 10 September 2010

(2a) Prove that for any $a,b,c,d \in \mathbb{R}$ $-(a-b) = b-a$

$x + (a-b) = 0 \quad (x \in \mathbb{R})$

Clearly $-(a-b)$ is a solution by F3

Let $(b-a) + (a-b) = 0$

$\Rightarrow [b+(-a)] + [a+(-b)] = 0$

$\Rightarrow b+(-a)+a+(-b) = 0$ by F1

$\Rightarrow b+(-a)+(-b)+a = 0$ commutativity

$\Rightarrow b+(-b)+(-a)+a = 0$ commutativity

$\Rightarrow [b+(-b)] + [(-a)+a] = 0$ by F1

$\Rightarrow 0 + 0 = 0$

$\Rightarrow (b-a)$ is also a solution

$\Rightarrow -(a-b) = b-a$ since the solution to the equation is unique QED

(2b) Prove that for any $a,b,c,d \in \mathbb{R}$, $(a-b)(c-d) = (ac+bd)-(ad+bc)$

$x - (a-b)(c-d) = 0 \quad (x \in \mathbb{R})$

Clearly $(a-b)(c-d)$ is a solution by F3

Let $[(ac+bd)-(ad+bc)] - (a-b)(c-d) = 0$

$\Rightarrow [(ac+bd)-(ad+bc)] - (a-b)\cdot c - (a-b)(-d) = 0$ distributive

$\Rightarrow [(ac+bd)-(ad+bc)] - (ac-bc) - [a(-d) - b(-d)] = 0$ distributive

$\Rightarrow [(ac+bd)-(ad+bc)] + (-1)(ac-bc) + (-1)[a(-1)d + (-1)b(-1)d] = 0$

$\Rightarrow [(ac+bd)+(-1)ad+(-1)bc] + (-1)ac + (-1)(-1)bc + (-1)a(-1)d + (-1)(-1)b(-1)d = 0$ distributive

$\Rightarrow ac + bd + (-ad) + (-bc) - ac + (1)bc + (1)ad + (1)(-1)bd = 0$ distributive, Property IV, commutativity

$\Rightarrow ac + bd + (-ad) + (-bc) + (-ac) + bc + ad + (-bd) = 0$ Property IV

$\Rightarrow [ac + (-ac)] + [bd + (-bd)] + [(-ad) + ad] + [(-bc) + bc] = 0$ commutative and F1

$\Rightarrow 0 + 0 + 0 + 0 = 0$

$\Rightarrow (ac+bd)-(ad+bc)$ is also a solution

$\Rightarrow (a-b)(c-d) = (ac+bd)-(ad+bc)$ since the solution to the equation is unique QED

(4a) Is 223/71 greater than 22/7?

$$223/71 - 22/7$$
$$= 223 \cdot \left(\tfrac{1}{71}\right) - 22\left(\tfrac{1}{7}\right)$$
$$= 223 \cdot \left(\tfrac{1}{71}\right) \cdot 1 - 22\left(\tfrac{1}{7}\right) \cdot 1$$
$$= 223 \cdot \tfrac{1}{71} \cdot 7 \cdot \tfrac{1}{7} - 22 \cdot \tfrac{1}{7} \cdot 71 \cdot \tfrac{1}{71}$$
$$= 223 \cdot 7 \cdot \tfrac{1}{71} \cdot \tfrac{1}{7} - 22 \cdot 71 \cdot \tfrac{1}{71} \cdot \tfrac{1}{7}$$
$$= (223 \cdot 7 - 22 \cdot 71)\left(\tfrac{1}{71} \cdot \tfrac{1}{7}\right)$$
$$= \underbrace{(1561 - 1562)}_{\notin \mathbb{R}_+ \cup \{0\}} \underbrace{\left(\tfrac{1}{71} \cdot \tfrac{1}{7}\right)}_{\in \mathbb{R}_+} \Rightarrow 223/71 - 22/7 \notin \mathbb{R}_+ \text{ by O5}$$

$\Rightarrow$ No, $223/71 - 22/7 \not> 22/7$

(4b) Is 265/153 greater than 1351/780

$$265/153 - 1351/780$$
$$= 265 \cdot \left(\tfrac{1}{153}\right) - 1351\left(\tfrac{1}{780}\right)$$
$$= 265 \cdot \left(\tfrac{1}{153}\right) \cdot 1 - 1351\left(\tfrac{1}{780}\right) \cdot 1$$
$$= 265 \cdot \tfrac{1}{153} \cdot 780 \cdot \tfrac{1}{780} - 1351 \cdot \tfrac{1}{780} \cdot 153 \cdot \tfrac{1}{153}$$
$$= 265 \cdot 780 \cdot \tfrac{1}{153} \cdot \tfrac{1}{780} - 1351 \cdot 153 \cdot \tfrac{1}{153} \cdot \tfrac{1}{780}$$
$$= (265 \cdot 780 - 1351 \cdot 153)\left(\tfrac{1}{153} \cdot \tfrac{1}{780}\right)$$
$$= \underset{\notin \mathbb{R}_+ \cup \{0\}}{(-3)} \underbrace{\left(\tfrac{1}{153} \cdot \tfrac{1}{780}\right)}_{\in \mathbb{R}_+} \Rightarrow 265/153 - 1351/780 \notin \mathbb{R}_+ \text{ by O5}$$

$\Rightarrow$ No, $265/153 \not> 1351/780$

(6) Show that if $a, b, x, y \in \mathbb{R}$ and $a < x < b$, $a < y < b$, then $|y-x| < b-a$

We will divide this proof into 2 parts:

Part I

Since $y < b$, $b-y \in \mathbb{R}_+$

Since $x > a$, $x-a \in \mathbb{R}_+$

$\Rightarrow (b-y)+(x-a) \in \mathbb{R}_+$

$\Rightarrow (b-y)+(x-a) > 0$

$\Rightarrow b-a+x-y > 0$

$\Rightarrow b-a+(-1)(-x+y) > 0$

$\Rightarrow b-a-(y-x) > 0$

$\Rightarrow b-a > y-x$

Part II

Since $x < b$, $b-x \in \mathbb{R}_+$

Since $y > a$, $y-a \in \mathbb{R}_+$

$\Rightarrow (b-x)+(y-a) \in \mathbb{R}_+$

$\Rightarrow (b-x)+(y-a) > 0$

$\Rightarrow b-a+y-x > 0$

$\Rightarrow b-a-[-(y-x)] > 0$

$\Rightarrow b-a > -(y-x)$

$$\Rightarrow b-a > \pm(y-x)$$

$$\Rightarrow b-a > |y-x|$$

or

$$|y-x| < b-a$$ QED

7a) Show that for any $a, b \in \mathbb{R}$, $\max\{a,b\} = \dfrac{a+b+|a-b|}{2}$

We will divide this proof into 3 cases:

Case I: $a < b$

Let $x - \max\{a,b\} = 0$

Clearly $\max\{a,b\}$ is a solution

$$\frac{a+b+|a-b|}{2} - \max\{a,b\}$$

since $a-b<0$

$$= \frac{a+b+[-(a-b)]}{2} - b$$

$$= \frac{a-a+b+b}{2} - b = \frac{2\cdot b}{2} - b$$

$$= \frac{1}{2}\cdot 2\cdot b - b = b - b = 0$$

$\Rightarrow \dfrac{a+b+|a-b|}{2}$ is also a solution

$$\Rightarrow \max\{a,b\} = \frac{a+b+|a-b|}{2}$$

Case II: $a > b$

Again, from Case I
$\max\{a,b\}$ is a solution to
$x - \max\{a,b\} = 0$

$$\frac{a+b+|a-b|}{2} - \max\{a,b\}$$

since $a-b>0$

$$= \frac{a+b+(a-b)}{2} - a$$

$$= \frac{a+a+b-b}{2} - a = \frac{2\cdot a}{2} - a$$

$$= \frac{1}{2}\cdot 2\cdot a - a = a - a = 0$$

$\Rightarrow \dfrac{a+b+|a-b|}{2}$ is also a solution

$$\Rightarrow \max\{a,b\} = \frac{a+b+|a-b|}{2}$$

Case III: $a = b$

Again from Case I,
$\max\{a,b\}$ is a solution to
$x - \max\{a,b\} = 0$

$$\frac{a+b+|a-b|}{2} - \max\{a,b\}$$

$$= \frac{a+a+|a-a|}{2} - \max\{a,a\}$$

$$= \frac{a+a+0}{2} - a = \frac{2\cdot a}{2} - a$$

$$= \frac{1}{2}\cdot 2\cdot a - a = a - a = 0$$

$\Rightarrow \dfrac{a+b+|a-b|}{2}$ is also a solution

$$\Rightarrow \max\{a,b\} = \frac{a+b+|a-b|}{2}$$

The results of all 3 cases $\Rightarrow \max\{a,b\} = \dfrac{a+b+|a-b|}{2}$ QED

7b) Show that for any $a, b \in \mathbb{R}$, $\min\{a,b\} = -\max\{-a,-b\} = \frac{a+b-|a-b|}{2}$

First, we will show that $\min\{a,b\} = -\max\{-a,-b\}$

Start with $x - \min\{a,b\} = 0$

Clearly $\min\{a,b\}$ is a solution

Case I: $a<b$

$\Rightarrow -a > -b$

$-\max\{-a,-b\} - \min\{a,b\}$

$= -(-a) - a$

$= a - a = 0$

$\Rightarrow -\max\{-a,-b\}$ is also a solution

Now consider

$\frac{a+b-|a-b|}{2} - \min\{a,b\}$

$= \frac{a+b-[-(a-b)]}{2} - a$ ← since $(a-b)<0$

$= \frac{a+b+(a-b)}{2} - a$

$= \frac{a+a+b-b}{2} - a = \frac{2 \cdot a}{2} - a$

$= \frac{1}{2} \cdot 2 \cdot a - a = a - a = 0$

$\Rightarrow \frac{a+b-|a-b|}{2}$ is also a solution

$\Rightarrow \min\{a,b\} = \max\{-a,-b\} = \frac{a+b-|a-b|}{2}$

Case II: $a>b$

$\Rightarrow -a < -b$

$-\max\{-a,-b\} - \min\{a,b\}$

$= -(-b) - b$

$= b - b = 0$

$\Rightarrow -\max\{-a,-b\}$ is also a sol'n

Now consider:

$\frac{a+b-|a-b|}{2} - \min\{a,b\}$

$= \frac{a+b-(a-b)}{2} - b$ ← since $(a-b)>0$

$= \frac{a-a+b+b}{2} - b$

$= \frac{2 \cdot b}{2} - b = \frac{1}{2} \cdot 2 \cdot b - b = b - b = 0$

$\Rightarrow \frac{a+b-|a-b|}{2}$ is also a solution

$\Rightarrow \min\{a,b\} = -\max\{-a,-b\} = \frac{a+b-|a-b|}{2}$

Case III: $a=b$

$\Rightarrow -a = -b$

$-\max\{-a,-b\} - \min\{a,b\}$

$= -\max\{-a,-a\} - \min\{a,a\}$

$= -(-a) - a$

$= a - a = 0$

$\Rightarrow -\max\{-a,-b\}$ is also a solution

Now consider:

$\frac{a+b-|a-b|}{2} - \min\{a,b\}$

$= \frac{a+a-|a-a|}{2} - \min\{a,a\}$

$= \frac{a+a-0}{2} - a$

$= \frac{2a}{2} - a = \frac{1}{2} \cdot 2 \cdot a - a = a - a = 0$

$\Rightarrow \frac{a+b-|a-b|}{2}$ is also a solution

$\Rightarrow \min\{a,b\} = -\max\{-a,-b\} = \frac{a+b-|a-b|}{2}$

The results of all 3 cases $\Rightarrow \min\{a,b\} = -\max\{-a,-b\} = \frac{a+b-|a-b|}{2}$

QED

(9a) Is the subset $\emptyset$ of $\mathbb{R}$ bounded from above or below?

By definition, an upper bound on a set $S \subset \mathbb{R}$ is a number $a \in \mathbb{R}$ such that $s \le a$ for each $s \in S$

Applying this definition to $\emptyset \subset \mathbb{R}$, we choose an $a' \in \mathbb{R}$

$\Rightarrow$ $s \le a'$ for all $s \in \emptyset$ since there are no elements in $\emptyset$

$\Rightarrow$ any $a' \in \mathbb{R}$ is an upper bound for $\emptyset$

$\Rightarrow$ $\emptyset \subset \mathbb{R}$ is bounded from above

Similarly, a lower bound on $S \subset \mathbb{R}$ is a number $a \in \mathbb{R}$ such that $s \ge a$ for each $s \in S$

For $\emptyset \subset \mathbb{R}$, we choose $a' \in \mathbb{R}$

$\Rightarrow$ $s \ge a'$ for all $s \in \emptyset$

$\Rightarrow$ any $a' \in \mathbb{R}$ is a lower bound for $\emptyset$

$\Rightarrow$ $\emptyset \subset \mathbb{R}$ is bounded from below

(9b) Does $\emptyset \subset \mathbb{R}$ have a l.u.b. or a g.l.b.?

For $\emptyset$ to have a l.u.b. y; 1) y must be an upper bound for $\emptyset$
2) if a is any upper bound for $\emptyset$, then $y \le a$

Suppose $\emptyset$ has a l.u.b. $y' \in \mathbb{R}$

Let $a' = y' - 1$

Since $a' \in \mathbb{R}$, it must be an upper bound for $\emptyset$ (from part a above)

But $y' > a'$ leading to a contradiction

$\Rightarrow$ $\emptyset$ does not have a l.u.b.

Similarly

Suppose $\emptyset$ has a g.l.b $y' \in \mathbb{R}$

Let $a' = y' + 1$

Since $a' \in \mathbb{R}$, it must be a lower bound for $\emptyset$ (from part a. above)

But $y' < a'$ leading to a contradiction

$\Rightarrow$ $\emptyset$ does not have a g.l.b.

10a) Find the g.l.b. and l.u.b. of $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$

This is the set $\{\frac{1}{n} : n \in \mathbb{N}\}$

When $n = 1$, $\frac{1}{n} = \frac{1}{1} = 1$

For any $n > 1$, say n', $n' > 1 \Rightarrow \frac{1}{n'} < \frac{1}{1} \Rightarrow$ $\underline{\underline{l.u.b = 1}}$

$n > 0 \Rightarrow 1/n > 0$ and by LUB 2 (p 26) $\Rightarrow$ $\underline{\underline{g.l.b. = 0}}$

10b) Find the g.l.b. and l.u.b of $\{\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{41}{80}, \cdots\}$

Note that this is the set $\left\{ \dfrac{\sum_{i=0}^{n-1} 3^i}{3^n} : n \in \mathbb{N} \right\}$

$$\frac{\sum_{i=0}^{n-1} 3^i}{3^n} = \frac{1}{3^n}\left(\frac{1-3^n}{1-3}\right) = -\frac{1}{2}\left(\frac{1-3^n}{3^n}\right) = -\frac{1}{2}\left(\frac{1}{3^n} - 1\right) = \frac{1}{2}\left(1 - \frac{1}{3^n}\right) = \frac{1}{2} - \frac{1}{2 \cdot 3^n}$$

For $n = 1$ and $n' > 1$, $n, n' \in \mathbb{N} \Rightarrow \frac{1}{3^n} = \frac{1}{3} > \frac{1}{3^{n'}} \Rightarrow \frac{1}{2 \cdot 3} > \frac{1}{2 \cdot 3^{n'}} \Rightarrow -\frac{1}{2 \cdot 3} < -\frac{1}{2 \cdot 3^{n'}}$

$\Rightarrow \frac{1}{2} - \frac{1}{2 \cdot 3} < -\frac{1}{2 \cdot 3^{n'}} \Rightarrow$ $\underline{\underline{g.l.b = \frac{1}{2} - \frac{1}{2 \cdot 3} = \frac{1}{3}}}$

Consider $n_1 < n_2 \Rightarrow 3^{n_1} < 3^{n_2} \Rightarrow 2 \cdot 3^{n_1} < 2 \cdot 3^{n_2} \Rightarrow \frac{1}{2 \cdot 3^{n_1}} > \frac{1}{2 \cdot 3^{n_2}}$

By LUB 2, $\frac{1}{2 \cdot 3^n}$ will approach $0 \Rightarrow$ $\underline{\underline{l.u.b = 1/2 - 0 = \frac{1}{2}}}$

10c) Find the g.l.b. and l.u.b. of $\{\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, + \cdots\}$

First note that $2 = \sqrt{4} = \sqrt{2+2} = \sqrt{2+\sqrt{2+2}} = \sqrt{2+\sqrt{2+\sqrt{2+2}}} = \cdots$

$\Rightarrow$ $\underline{\underline{l.u.b. = 2}}$

Then note that $2 < 2 + \sqrt{2} \Rightarrow \sqrt{2} < \sqrt{2+\sqrt{2}}$ and $2 < 4 \Rightarrow \sqrt{2} < \sqrt{4} \Rightarrow \sqrt{2} < 2$

Since each term in the expansion of the set adds $\epsilon > 0$

$\Rightarrow$ $\underline{\underline{g.l.b. = \sqrt{2}}}$

(11) Prove that if $a \in \mathbb{R}$, $a > 1$, then the set $\{a, a^2, a^3, \ldots\}$ is not bounded from above

Since $a > 1 \Rightarrow a - 1 > 0$;

Let $\varepsilon = a - 1 \Rightarrow \varepsilon > 0, \varepsilon \in \mathbb{R}$

By LUB 2 (p 26), there exists $n \in \mathbb{N}$, such that $\frac{1}{n} < \varepsilon$

Choose such an $n \in \mathbb{N}$

$\Rightarrow a > 1 + \frac{1}{n}$

$\Rightarrow a^n > (1 + \frac{1}{n})^n$

When $n = 1$, $(1 + \frac{1}{n})^n = 2$

By the binomial expansion, we know $(1+\frac{1}{n})^n = \sum_{k=0}^{n} \binom{n}{k} (1)^{n-k} (\frac{1}{n})^k$

$$= \binom{n}{0}(1)^n(\tfrac{1}{n})^0 + \binom{n}{1}(1)^{n-1}(\tfrac{1}{n})^1 + \sum_{k=2}^{n}\binom{n}{k}(1)^{n-k}(\tfrac{1}{n})^k = 1 + 1 + \underbrace{\sum_{k=2}^{n}\binom{n}{k}(\tfrac{1}{n})^k}_{\in \mathbb{R}_+} > 2 \quad \forall n > 1$$

$\Rightarrow a^n > (1 + \frac{1}{n})^n \geq 2$

Assume to the contrary that the set $\{a^n : n \in \mathbb{N}\}$ is bounded $\Rightarrow$ l.u.b. exists. Let u = l.u.b. of the set

$\Rightarrow u > a^n \; \forall n \in \mathbb{N}$

From above $(a^n)^m > 2^m \Rightarrow a^{nm} > 2^m \quad (m \in \mathbb{N})$

Let m' be the largest m such that $2^{m'} < u$

Since $m' \in \mathbb{N}$, $m'+1 \in \mathbb{N} \Rightarrow 2^{m'+1} > u$

$\Rightarrow (a^n)^{m'+1} > 2^{m'+1} > u$ which is a contradiction

$\Rightarrow$ The set $\{a^n : n \in \mathbb{N}\}$ is unbounded

(12) Let X, Y be nonempty subsets of $\mathbb{R}$ whose union is $\mathbb{R}$ and such that each element of X is less than each element of Y. Prove that there exists $a \in \mathbb{R}$ such that X is one of the two sets

$$\{x \in \mathbb{R} : x \leq a\} \quad \text{or} \quad \{x \in \mathbb{R} : x < a\}$$

First, we claim that $X \cap Y = \emptyset$

To see this, assume, to the contrary that $X \cap Y$ is nonempty

Then there exists $x' \in \mathbb{R}$ such that $x' \in X$ and $x' \in Y$

But then, by assumption, $x' < x'$ which is a controdiction

Thus $a \in \mathbb{R}$ must either be in X or Y but not both

Next construct $a = \text{l.u.b. } X$, which we must first prove exists:

By assumption, X is nonempty

Also, $\forall x \in X$, there exists a $y \in Y$, such that $x < y$ (for all y, in fact)

$\Rightarrow X$ is bounded from above

$\Rightarrow X$ has a l.u.b. (Property VII)

Case I: $a \in X$

$\Rightarrow \forall x \in X,\ x \leq a$, (since $a = \max\{X\}$)

$\Rightarrow \forall x \in X, y \in Y;\ x \leq a < y$ (since $\forall x \in X, y \in Y$, $x < y$ <u>and</u> $a \in X \Rightarrow a < y$)

$\Rightarrow X = \{x \in \mathbb{R} : x \leq a\}$ (since $X \cup Y = \mathbb{R}$)

Case II: $a \in Y$

$\Rightarrow \forall x \in X,\ x < a$ (since $a \in Y$ and $\forall y \in Y, x \in X,\ x < y$)

$\Rightarrow \forall x \in X, y \in Y\ \ x < a \leq y$ (since $\forall y \in Y; y \neq a, y > a$; otherwise $y < a$ would be in X, which is a contradiction)

$\Rightarrow X = \{x \in \mathbb{R} : x < a\}$ QED

(13) If S_1, S_2 are non-empty subsets of $\mathbb{R}$ that are bounded from above, prove that

$$l.u.b.\{x+y : x \in S_1, y \in S_2\} = l.u.b.\, S_1 + l.u.b.\, S_2$$

Since $S_1 \neq \phi$ and $S_1 \subset \mathbb{R}$ and S_1 is bounded above

$\Rightarrow$ S_1 has a l.u.b.

Let $l.u.b.\, S_1 = a$

Similarly, since $S_1 \neq \phi$, $S_1 \subset \mathbb{R}$, and S_1 is bounded above

$\Rightarrow$ S_2 has a l.u.b.

Let $l.u.b.\, S_2 = b$

By the proof in the "Aside" block, we can choose $x' \in S_1$ such that $x' \geq a - \varepsilon/2$ where $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$ (assuming $a \notin S_1$)

Similarly, we can choose $y' \in S_2$ such that $y' \geq b - \varepsilon/2$ (Note that we are choosing 1 ε so that both inequalities are true) (assuming $b \notin S_2$)

Call the set $Z = \{x+y : x \in S_1, y \in S_2\}$

We also know that since

1) for all $x \in S_1$, $x \leq a$ and
2) for all $y \in S_2$, $y \leq b$

Then $x + y \leq a + b$

$\Rightarrow$ Z is bounded above

Now let $z' = x' + y'$

$\Rightarrow$ $z' \in Z$ since $x' \in S_1$ and $y' \in S_2$

$\Rightarrow z' = x' + y' \geq a - \varepsilon/2 + b - \varepsilon/2 = a + b - \varepsilon$

Thus we can choose z' to be arbitrarily close to $a+b$

$\Rightarrow l.u.b.\, Z = l.u.b.\{x+y : x \in S_1, y \in S_2\} = a + b = l.u.b.\, S_1 + l.u.b.\, S_2$ QED

Aside

Consider a non-empty set $S \subset \mathbb{R}$ with $l.u.b.\, S = y$, $x \in S$, $y \notin S$

Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$

Construct $x' = y - \varepsilon$ and assume that ε is chosen such that $y - \varepsilon > g.l.b.\, S$, if it exists. We can do this since S is non-empty and ε can be chosen arbitrarily small to guarantee it.

$\Rightarrow x' < y$

$\Rightarrow x' \in S$ since it is $>$ g.l.b. and is not the l.u.b. (since the l.u.b is unique)

Since $x' = y - \varepsilon \Rightarrow x' \geq y - \varepsilon$

$\Rightarrow \exists x \in S, \varepsilon \in \mathbb{R}, \varepsilon > 0$, such that $x \geq y - \varepsilon$

We can get arbitrarily close to y in S even though $y \notin S$.

<u>Note</u>: We proved the least trivial case, when $a \notin S_1$ and $b \notin S_2$. The proof can be repeated by replacing $\varepsilon/2$ with zero and inequalities with equalities in the statements involving x', y' and z'.

(16) Decimal (10-nary) expansions of real numbers were defined by special reference to the number 10. Show that real numbers have b-nary expansions with analogous properties, where b is any interger greater than 1.

Let $b \in \mathbb{N}, b>1$

If $a_0 \in \mathbb{Z}, n \in \mathbb{N}$, and $a_1, a_2, \ldots, a_n$ any integers chosen from $0,1,2,\ldots,b$, the symbol $a_0 . a_1 a_2 \ldots a_n$ will mean the rational number

$$a_0 + \frac{a_1}{b^1} + \frac{a_2}{b^2} + \cdots + \frac{a_n}{b^n}$$

Note here why $b>1$, since each fraction in the series must be less than 1.

If m is a positive integer less than n, then

$$a_0 . a_1 \ldots a_m \le a_0 . a_1 \ldots a_n = a_0 . a_1 \ldots a_m + a_{m+1} \cdot b^{-(m+1)} + \cdots + a_n \cdot b^{-n}$$
$$\le a_0 . a_1 \ldots a_m + (b-1) \cdot b^{-(m+1)} + \cdots + (b-1) \cdot b^{-n}$$

After adding 10^{-n} to the last number and cancelling, we get

$$a_0 . a_1 \ldots a_m \le a_0 . a_1 \ldots a_n < a_0 . a_1 \ldots a_m + b^{-m}$$

As before with base 10, the infinite b-mal (since the term "decimal" implies 10) is

$$a_0 . a_1 a_2 a_3 \ldots$$

Again, the set $\{a_0 . a_1 a_2 \ldots a_n : n \in \mathbb{Z}_+\}$ is non-empty and bounded above (for any integer $m > \emptyset$, $a_0 . a_1 a_2 \ldots a_m + b^{-m}$ is an upper bound)

$$\Rightarrow a_0 . a_1 a_2 a_3 \ldots = \text{l.u.b.} \{a_0 . a_1 \ldots a_n : n = \text{positive integer}\}$$

and for any positive integer n, we have the inequality

$$a_0 . a_1 \ldots a_n \le a_0 . a_1 a_2 a_3 \ldots \le a_0 . a_1 \ldots a_n + b^{-n}$$

And, again, any real number is represented be at least one infinite b-mal. To see this, apply LUB4 to the case $N = b^m$, where $m \in \mathbb{Z}_+$ we get

$$a_0 . a_1 \ldots a_m \le x < a_0 . a_1 \ldots a_m + b^{-m}$$

For multiplication and addition, we would have to be careful to consider the new base b when "carrying digits", but otherwise the procedures would not change.

$\Rightarrow$ real numbers have b-nary expansions with analogous properties to decimals, where b is any integer greater than 1.