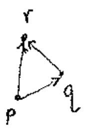
MATH 4317 Real. Anal. | HW # 3 Due: 9-15 | Brian Beck

Problems Ch3: 1, ~~2~~, 5, 6, ~~7~~, ~~10~~, ~~11~~, 12, 13, 14 (p61-62)

Group Work. Brian Beck. Zhongyuan Yu. check 1, 5, 13 in white sheets

1

a) n-tuples of $\mathbb{R}$ $(x_1 \dots x_n)$ $(y_1 \dots y_n)$ $d((x_1 \dots x_n),(y_1 \dots y_n)) = \sum_{i=1}^{n} |x_i - y_i|$



① let $(x_1 \dots x_n) = p$ and $(y_1 \dots y_n) = q$ $d(p,q) \geq 0$ ✓
b/c $|x_i - y_i| \geq 0$ and $\sum_{i=1}^{n}$ of positive #'s is positive

② $\sum_{i=1}^{n} |x_i - y_i| = 0$ iff $x_i = y_i$ $\forall i$; b/c $|x_i - y_i| \geq 0$

thus $d(p,q) = 0$ iff $p = q$ ✓

③ $d(p,q) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |y_i - x_i| = d(q,p)$ ✓

④ $d(p,r) = \sum_{i=1}^{n} |x_i - r_i|$ $d(p,q) = \sum_{i=1}^{n} |x_i - y_i|$

$d(q,r) = \sum_{i=1}^{n} |y_i - r_i|$ Note: $|a+b| \leq |a| + |b|$

So $|x_i - r_i| \leq |x_i - y_i| + |y_i - r_i|$ $\forall i$ (general Δ-inequality)

Thus $\sum_{i=1}^{n} |x_i - r_i| \leq \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - r_i|$ ✓

b) $x = (x_1 \dots x_n \dots)$ x has an UB $\longrightarrow$ x has a LUB.

① $d(x,y) = LUB\{z_1, z_2, z_3 \dots\}$ where each $z_i \geq 0$, Thus $d(x,y) \geq 0$ ✓

② $z_1 = |x_1 - y_1| = 0$ iff $x_1 = y_1$, $z_i = 0$ iff $x_i = y_i$,
Thus $LUB\{z_1, z_2 \dots\} = 0$ iff $x_i = y_i = z_i = 0$ $\forall i$ ✓

③ $|x_i - y_i| = |y_i - x_i| = z_i$, so $d(x,y) = d(y,x)$ ✓
$LUB\{|x_1 - y_1|, |x_2 - y_2| \dots\} = LUB\{|y_1 - x_1|, |y_2 - x_2| \dots\}$

④ as in part a), $|x_i - r_i| \leq |x_i - y_i| + |y_i - r_i|$ $\forall i$
$|x_i - r_i| = |(x_i - y_i) + (y_i - r_i)| \leq |x_i - y_i| + |y_i - r_i|$

and so $LUB\{|x_1 - r_1| \dots\} \leq LUB\{|x_1 - y_1| \dots\} + LUB\{|y_1 - r_1| \dots\}$ ✓

c) $(E_1 \times E_2, d)$ $(E_1, d_1), (E_2, d_2)$ are ms, $d((x_1 x_2)(y_1 y_2)) = MAX\{d_1(x_1 y_1), d_2(x_2 y_2)\}$

① $MAX\{d_1, d_2\} \geq 0$ b/c $d_1 + d_2 \geq 0$ ✓
② $MAX\{d_1, d_2\} = 0$ iff $d_1, d_2 = 0$, occurs iff $x_1 = y_1$, $y_2 = x_2$ ✓
③ $MAX\{d_1 d_2\} = MAX\{d_2 d_1\}$ by def. ✓

④ Show $MAX\{d_1(x_1 r_1)\ d_2(x_2 r_2)\} \leq MAX\{d_1(x_1 y_1)\ d_2(x_2 y_2)\} + MAX\{d_1(y_1 r_1)\ d_2(y_2 r_2)\}$
Given: $d_1(x_1 r_1) \leq d_1(x_1 y_1) + d_1(y_1 r_1)$
$d_2(x_2 r_2) \leq d_2(x_2 y_2) + d_2(y_2 r_2)$

Suppose $d_1(x_1 r_1) > d_2(x_2 r_2)$

Then $d_2(x_2 r_2) \le d_1(x_1 y_1) + d_1(y_1 r_1)$ and

$d_1(x_1 r_1) \le MAX\{d_1(x_1 y_1)\, d_2(x_2 y_2)\} + MAX\{d_1(y_1 r_1)\, d_2(y_2 r_2)\}$ ✓

and vice versa for $d_2(x_2 r_2) > d_1(x_1 r_1)$

$\boxed{2}$ $(\mathbb{R}\times\mathbb{R}, d)$ $\quad d((xy)(x'y')) = \begin{cases} |y| + |y'| + |x-x'| & \text{if } x \ne x' \\ |y-y'| & \text{if } x = x' \end{cases}$

① $d(p,q) \ge 0$

b/c all terms use absolute value ✓

② $d(pq) = 0$ iff $x=x'$ AND $y=y'$ ✓ $|y-y'| = 0$ when $y=y'$

③ $d(pq) = d(qp)$ b/c $|y| + |y'| + |x-x'| = |y'| + |y| + |x'-x|$

AND $|y-y'| = |y'-y|$

④

Show $d((xy)(x'y')) \le d((xy)(pq)) + d((pq)(x'y'))$

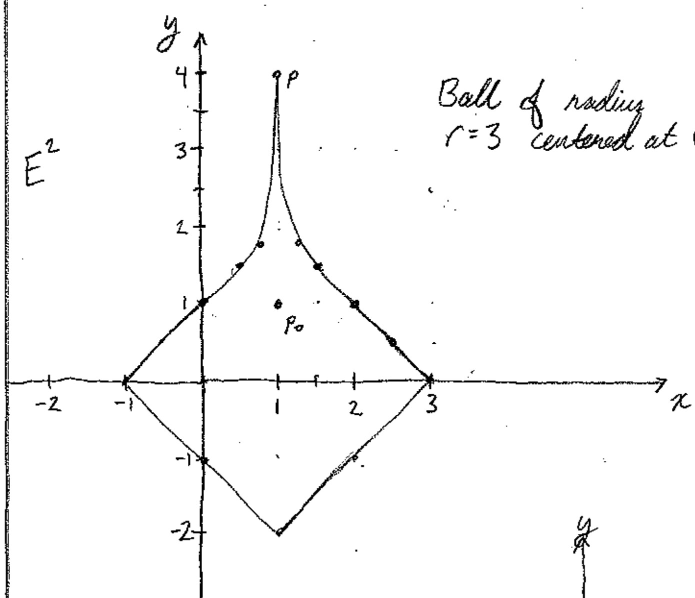
if $x=x'$ this has already been proven. assume $x \ne x'$

show: $\cancel{|y|} + \cancel{|y'|} + |x-x'| \le (\cancel{|y|} + |q| + |x-p|) + (|q| + \cancel{|y'|} + |p-x'|)$

$|x-x'| \le 2|q| + |x-p| + |p-x'|$

$|x-x'| = |(x-p) + (p-x')| \le |x-p| + |p-x'|$ SO

$\underline{|x-x'| \le 2|q| + |x-p| + |p-x'|}$ ✓

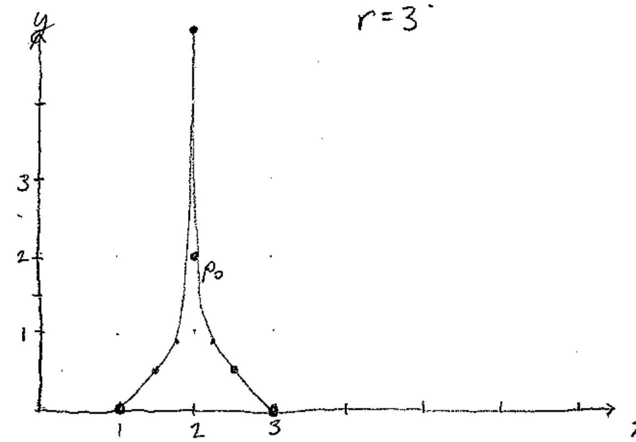


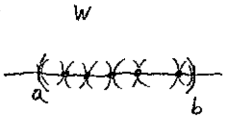
Ball of radius $r=3$ centered at $(1,1)$

Open Ball $r=3$ $p_0 = (3,2)$

$|y| + |y'| + |x-x'| < 3$

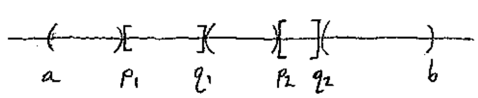
$r=3$



5 Bounded open subset of $\mathbb{R}$ is the union of disjoint open intervals 

$S \subset (a,b)$ an open subset w GLB a and LUB b

Since S is open, $\forall\ p_0 \in S\ \exists$ an open ball of radius r centered at p_0
also, since S is open, any interval $[p\,q] \notin S$ where $a < p < q < b$ is closed

like this : 

if $[p\,q] \notin S$, then $S = (a\,p) \cup (q\,b)$

reason: if S is open, S^c is closed $([p\,q] \in S^c)$

Thus $S = (a\,p_1) \cup (q_1\,p_2) \cup (q_2\,p_3) \cup (q_3\,p_4) \cup \ldots (p_n\,b)$

$S = (a\,p_1) \cup \left[\bigcup_{i=1}^{n} (q_i\,p_{i+1})\right] \cup (p_{n+2}\,b)$

so S is a union of open intervals that are disjoint ✓

6 $S = \{ (x_1 x_2) \in E^2 : x_1 x_2 = 1\ \ x_1 > 0 \}$ is closed

a set is closed if its compliment is open

$S^c = \{ (x_1 x_2) \in E^2 : \ x_1 x_2 \neq 1$ OR $x_1 \leq 0 \}$ < 0 b/c if $x_1 = 0$, $x_1 x_2 \neq 1$

SO $S^c = A \cup B$, $A = \{ x_1 x_2 \neq 1 \}$, $B = \{ x_1 < 0 \}$

B is an open set ✓

✓ assuming the usual distance formula for E^2, A is an open set
b/c $\forall\ x_i x_j \in A$, $\exists$ a ball ST $B((x_i x_j), r) \in A$

thus S^c is open , b/c S^c is a union of open sets.

$\therefore$ S is closed

7 A set ($\neq \emptyset$, closed) bounded from below has a least element ← show

let $a = GLB(S)$, if $a \notin S$, then $a \in S^c$
S^c is an open set, so there $\exists$ an $\varepsilon \in \mathbb{R}$ ST an open ball centered at a, : $B(a,\varepsilon) \in S^c$
thus $a+\varepsilon \in S^c$ and $a+\varepsilon$ is a lower bound for S ← contradicts assumption that $a = GLB(S)$

$\therefore$ $a = GLB(S) \in S$ and a is the least element of S

10 Given $\lim_{n\to\infty} p_n = p$, Show $\{p, p_1, p_2 \ldots\}$ is <u>closed</u> $= S$

since the set S contains the point $p = \lim_{n\to\infty} p_n \in S$, S must be closed by the theorem on p 47-48:

Suppose S is not closed $\therefore \longrightarrow$ S^c is not open
Then $\exists$ $p \in S^c$ ST $B(p,r)$ contains points in S
Then can choose $p_n \in S$ ST $d(p,p_n) < 1/n$
$\lim_{n\to\infty} p_n = p$ with <u>$p \notin S$</u> ← <u>CONTRADICTION b/c $p \in S$!</u>

11 Show $\lim_{n\to\infty} \dfrac{\sum_{i=1}^{n} a_i}{n} = a$ <u>given:</u> a_i's $\to a$. ie, $\exists$ $\varepsilon > 0$ ST $|a - a_i| < \varepsilon$ for $n > N$

Split the summation: $\lim_{n\to\infty}\left[\dfrac{\sum_{i=1}^{N} a_i + \sum_{i=N+1}^{n} a_i}{n}\right] = \lim_{n\to\infty}\left[\dfrac{\sum_{i=1}^{N} a_i}{n}\right] + \lim_{n\to\infty}\left[\dfrac{\sum_{i=N+1}^{n} a_i}{n}\right]$ ← $= 0$

since a_i's $\to a$, $\therefore$

$$\sum_{i=N+1}^{n}(a-\varepsilon) \le \sum_{i=N+1}^{n} a_i \le \sum_{i=N+1}^{n}(a+\varepsilon)$$

$$(n-N-1)(a-\varepsilon) \le \sum_{i=N+1}^{n} a_i \le (n-N-1)(a+\varepsilon)$$

$$\frac{(n-N-1)(a-\varepsilon)}{n} \le \frac{\sum_{i=N+1}^{n} a_i}{n} \le \frac{(n-N-1)(a+\varepsilon)}{n}$$

let $n \to \infty$, $\quad 1\cdot(a-\varepsilon) \le \lim_{n\to\infty}\left[\dfrac{\sum_{i=N+1}^{n} a_i}{n}\right] \le 1\cdot(a+\varepsilon)$

since we can choose ε as small as we want, $\lim_{n\to\infty} \dfrac{\sum a_i}{n} \longrightarrow a$ ✓

12 Prove $x_1\, x_2 \ldots$ is <u>UN-bounded</u> $\quad x_1 = 1 \quad x_{n+1} = x_n + \frac{1}{x_n^2} \quad = \{1, 2, 2\tfrac{1}{4}, \ldots \ldots\}$

Assume the set $S = \{x_1 x_2 \ldots\}$ is bounded. + find a contradiction.

if S is bounded then $x_n \le B \quad \forall n \quad (B > 2)$ S is an <u>increasing</u> sequence.

$x_{n+1} = x_n + \frac{1}{x_n^2} \ge x_n + \frac{1}{B^2} \quad$ (b/c $1/B^2 \le \frac{1}{x_n^2}$) true $\forall n$

So $x_2 \ge x_1 + \frac{1}{B^2}$, $\quad x_3 \ge x_2 + \frac{1}{B^2} = x_1 + \frac{1}{B^2} + \frac{1}{B^2} = x_1 + \frac{2}{B^2}$

Thus $x_n \ge x_1 + \frac{n-1}{B^2}$ $\quad \leftarrow$ but as $n \to \infty$, $x_n \to \infty$
so S must be un-bounded

13 Sequence is $x_n = \frac{1}{2 + x_{n-1}} \quad x_1 = 1/2 \quad X = \{1/2, 2/5, 5/12, 12/29, 29/70 \ldots$

Show that the sequences of alternate terms are monotonic.
First look @ the <u>odd</u> terms x_{2m+1}

given x_n <u>(assume n odd)</u>: $x_{n+1} = \frac{1}{2 + x_n} \quad x_{n+2} = \frac{1}{2 + \frac{1}{2+x_n}} = \frac{x_n + 2}{2x_n + 5} \quad x_{n+3} = \frac{2x_n + 5}{5x_n + 12}$

$(2+x_n)(2+\frac{1}{2+x_n})$

$2 + x_n + 1$

$x_{2m+1} = \frac{x_{2m-1} + 2}{2x_{2m-1} + 5}$ $\quad$ (odd terms, monotonically decreasing, $x_1 = 1/2 \quad x_3 = 5/12, \quad x_5 = 29/70 \ldots$)
Why? given x_n: any odd term,
$x_{n+2} = \frac{x_n + 2}{2x_n + 5} \le x_n \quad \forall n$

and $\lim_{m\to\infty}(x_{2m+1}) = X$, $\longrightarrow$ solve $X = \frac{2+X}{2X+5}$ $\quad X = \sqrt{2} - 1$

<u>look @ the even terms:</u>

$x_{n+1} = 2\, x_n$

$X = 2\, X$

$x_{2m} = \frac{x_{2m-2} + 2}{2x_{2m-2} + 5}$ $\quad$ where $x_2 = 2/5 \quad x_4 = 12/29 \quad x_6 = 70/169$ monotonically <u>increasing</u>

Why? $\frac{1}{2 + x_n} \le \frac{2x_n + 5}{5x_n + 12}$ when x_n is any <u>odd</u> term

and $\lim_{m\to\infty}(x_{2m}) \longrightarrow$ solve $X = \frac{X+2}{2X+5}$, $\quad X = \sqrt{2} - 1$

$\frac{1}{2 + x_n}$

an increasing + decreasing subsequence converge to the same limit, thus the sequence converges with limit $\sqrt{2} - 1$!

$\boxed{14}$ Any sequence in $\mathbb{R}$ has a monotonic subsequence. $S = \{x_1, x_2 \ldots\}$

① suppose the subsequence has no least term.
then there always exists a b_{k+1}. ST $b_{k+1} \le b_k$ in the subsequence.
thus the subsequence is monotonically decreasing.

② the subsequence has a least term, then
the subsequence b_k has a GLB.

construct the following subsequence:

$b_k = GLB\{x_n : n \ge k\}$ where x_n are the terms of the original sequence.

here $b_{k+1} \ge b_k$, b/c the GLB of the sequence can only increase as we take away terms.

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1. Verify that the following are metric space

1 (a) all n-tuples of real numbers, with $d((x_1,\cdots,x_n),(y_1,\cdots,y_n)) = \sum_{i=1}^{n} |x_i - y_i|$

(1) $d(p,q) \geq 0$ for all $p, q \in E$

$d((x_1,\cdots,x_n),(y_1,\cdots,y_n)) = |x_1-y_1| + |x_2-y_2| + \cdots + |x_n-y_n| \geq 0$

(2) $d(p,q) = 0$ iff $p=q$

$d((x_1,\cdots,x_n),(y_1,\cdots,y_n)) = |x_1-y_1| + |x_2-y_2| + \cdots + |x_n-y_n| = 0$

$\Leftrightarrow x_1=y_1, x_2=y_2, \cdots, x_n=y_n \Leftrightarrow x=y$

(3) $d(p,q) = d(q,p)$ for all $p, q \in E$

$d((x_1,\cdots,x_n),(y_1,\cdots,y_n)) = |x_1-y_1| + |x_2-y_2| + \cdots + |x_n-y_n|$

$= |y_1-x_1| + |y_2-x_2| + \cdots + |y_n-x_n| = d((y_1,\cdots,y_n),(x_1,\cdots,x_n))$

(4) $d(p,r) \leq d(p,q) + d(q,r)$ for all $p, q, r \in E$

$x=(x_1,\cdots,x_n)$ $y=(y_1,\cdots,y_n)$ $z=(z_1,\cdots,z_n)$

$d((x_1,\cdots,x_n),(y_1,\cdots,y_n)) = |x_1-y_1| + |x_2-y_2| + \cdots + |x_n-y_n|$

$= |(x_1-z_1)+(z_1-y_1)| + |(x_2-z_2)+(z_2-y_2)| + \cdots + |(x_n-z_n)+(z_n-y_n)|$

$\leq |x_1-z_1| + |z_1-y_1| + |x_2-z_2| + |z_2-y_2| + \cdots + |x_n-z_n| + |z_n-y_n|$

$= (|x_1-z_1| + |x_2-z_2| + \cdots + |x_n-z_n|) + (|z_1-y_1| + |z_2-y_2| + \cdots + |z_n-y_n|)$

$= d((x_1,\cdots,x_n),(z_1,\cdots,z_n)) + d((z_1,\cdots,z_n),(y_1,\cdots,y_n))$

1 (b) all bounded infinite sequence $x=(x_1,x_2,\cdots)$ of elements of $\mathbb{R}$ with

$d(x,y) = \text{l.u.b}\{|x_1-y_1|, |x_2-y_2|, \cdots\}$

(1) $d(p,q) \geq 0$ for all $p, q \in E$

$d(x,y) = \text{l.u.b}\{|x_1-y_1|, |x_2-y_2|, \cdots\} \geq |x_a-y_a| \geq 0$ $\quad \forall a \in \{1,2,\cdots\}$

(2) $d(p,q)=0$ iff $p=q$

$0 = d(x,y) = \text{l.u.b}\{|x_1-y_1|, |x_2-y_2|, \cdots\} \geq |x_a-y_a|$ $\quad \forall a \in \{1,2,\cdots\}$

$\Leftrightarrow |x_a-y_a| = 0 \quad \forall a \in \{1,2,\cdots\} \Leftrightarrow x_1=y_1, x_2=y_2, \cdots \Leftrightarrow x=y$

(3) $d(p,q) = d(q,p)$ for all $p, q \in E$

$d(x,y) = \text{l.u.b}\{|x_1-y_1|, |x_2-y_2|, \cdots\} = \text{l.u.b} = \{|y_1-x_1|, |y_2-x_2|, \cdots\} = d(y,x)$

1(b) i4) $d(p,r) \le d(p,q) + d(q,r)$ for all $p, q, r \in E$

$d(x,y) = \text{l.u.b}\{|x_1-y_1|, |x_2-y_2|, \cdots\} = \text{l.u.b}\{|x_1-z_1+z_1-y_1|, |x_2-z_2+z_2-y_2|, \cdots\}$

$\le \text{l.u.b}\{|x_1-z_1|+|z_1-y_1|, |x_2-z_2|+|z_2-y_2|, \cdots\}$

$= \text{l.u.b}\{|x_1-z_1|, |x_2-z_2|, \cdots\} + \text{l.u.b}\{|z_1-y_1|, |z_2-y_2|, \cdots\} = d(x,z)+d(z,y)$

1(c) $(E_1 \times E_2, d)$ where (E_1, d_1) (E_2, d_2) are metric spaces and d is given by

$d((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}$

i1) $d(p,q) \ge 0 \quad \forall\, p, q \in E$

$d((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\} \ge d_a(x_a,y_a) \ge 0 \quad \forall a \in \{1,2\}$

i2) $d(p,q) = 0$ iff $p=q$

$0 = d((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\} \ge d_a(x_a,y_a) \quad \forall a \in \{1,2\}$

$\Leftrightarrow d_1(x_1,y_1)=0$ and $d_2(x_2,y_2)=0 \quad \Leftrightarrow \quad x_1=y_1, x_2=y_2 \quad \Leftrightarrow X=Y$

(E_1,d_1) metric space $\Rightarrow d_1(x_1,y_1)=0$ iff $x_1=y_1$

(E_1,d_2) metric space $\Rightarrow d_2(x_2,y_2)=0$ iff $x_2=y_2$

i3) $d(p,q) = d(q,p) \quad \forall\, p, q \in E$

$d((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}$

$= \max\{d_1(y_1,x_1), d_2(y_2,x_2)\} = d((y_1,y_2),(x_1,x_2))$

i4) $d(p,r) \le d(p,q) + d(q,r)$ for all $p, q, r \in F$

$d((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}$

same for $d((x_1,x_2),(z_1,z_2)) = \max\{d_1(x_1,z_1), d_2(x_2,z_2)\}$

$d((z_1,z_2),(y_1,y_2)) = \max\{d_1(z_1,y_1), d_2(z_2,y_2)\}$

(E_1, d_1) matric space $\Rightarrow d(x_1,y_1) \le d(x_1,z_1) + d(z_1,y_1)$

(E_2, d_2) matric space $\Rightarrow d(x_2,y_2) \le d(x_2,z_2) + d(z_2,y_2)$

if $d_1(x_1,y_1) \ge d_2(x_2,y_2)$

$d((x_1,x_2),(y_1,y_2)) = d_1(x_1,y_1) \le d_1(x_1,z_1) + d_1(z_1,y_1) \le \max\{d_1(x_1,z_1), d_2(x_2,z_2)\}$

$+ \max\{d_1(z_1,y_1), d_2(z_2,y_2)\} = d((x_1,x_2),(z_1,z_2)) + d((z_1,z_2),(y_1,y_2))$

same for $d_1(x_1,y_1) \le d_2(x_2,y_2)$

$d((x_1,x_2),(y_1,y_2)) \le d((x_1,x_2),(z_1,z_2)) + d((z_1,z_2),(y_1,y_2))$

5. Prove that any bounded open subset $\mathbb{R}$ is the union of disjoint open intervals

define $S = \bigcup_{i \in I} B_{\varepsilon_i}(p_i)$ where $B_{\varepsilon_i}(p_i) = (p_i - \varepsilon_i, p_i + \varepsilon_i)$

as a bounded open set which is the union of any collection of open subset

if two interval $B_{\varepsilon_i}(p_i) \cap B_{\varepsilon_j}(p_j) \neq \emptyset$ $i \neq j$

then we have $B_{\varepsilon_k}(p_k) = (\min(p_i - \varepsilon_i, p_j - \varepsilon_j), \max(p_i + \varepsilon_i, p_i + \varepsilon_j))$

$\varepsilon_k = \frac{1}{2} |\max\{p_i + \varepsilon_i, p_j + \varepsilon_j\} - \min\{p_i - \varepsilon_i, p_j - \varepsilon_j\}|$

$p_k = \max\{p_i + \varepsilon_i, p_j + \varepsilon_j\} - \varepsilon_k$

and this $B_{\varepsilon_k}(p_k)$ can replace $B_{\varepsilon_i}(p_i)$ and $B_{\varepsilon_j}(p_j)$

thus we can make any two intervals $B_{\varepsilon_i}(p_i) \cap B_{\varepsilon_j}(p_j) = \emptyset$ $\forall i \neq j$

by repeating the process

therefore proves any bounded open set of $\mathbb{R}$ is the union of disjoint open intervals.

13. consider the sequence of real numbers $\frac{1}{2}, \frac{1}{2+\frac{1}{2}}, \frac{1}{2+\frac{1}{2+\frac{1}{2}}}, \cdots$

show that this sequence is convergent and find its limit by first showing that the two sequence of alternative terms are monotonic and finding their limits

$f(1) = \frac{1}{2} \qquad f(n) = \frac{1}{2+f(n-1)} \qquad n = 2, 3, \cdots$

odd terms:

$$f(1) - f(3) = \frac{1}{2} - \frac{1}{2+\frac{1}{2+\frac{1}{2}}} = \frac{1}{2} - \frac{5}{12} = \frac{1}{12} > 0$$

assume $f(2n+1) - f(2n+3) > 0$

$$f(2n+3) - f(2n+5) = \frac{1}{2+f(2n+2)} - \frac{1}{2+f(2n+4)} = \frac{1}{2+\frac{1}{2+f(2n+1)}} - \frac{1}{2+\frac{1}{2+f(2n+3)}}$$

$$= \frac{f(2n+1)+2}{2f(2n+1)+5} - \frac{f(2n+3)+2}{2f(2n+3)+5} = \frac{f(2n+1) - f(2n+3)}{[2f(2n+1)+5][2f(2n+3)+5]} > 0$$

$\therefore$ $f(1), f(3), f(5), \cdots$ is decreasing and thus is monotonic.

assume that exist N, $|f(2n+1) - a| < \varepsilon$ for $n > N \Rightarrow \lim_{n\to\infty} f(2n+1) = a$

$$\lim_{n\to\infty} f(2n+3) = \lim_{n\to\infty} \frac{1}{2+f(2n+1)} = \frac{1}{2+a} = a \Rightarrow a = \sqrt{2}-1 \quad (-\sqrt{2}-1<0 \text{ delete})$$

check $f(1) = \frac{1}{2} > \sqrt{2}-1$

assume $f(2n+1) > \sqrt{2}-1 \Rightarrow f(2n+3) = \frac{1}{2+\frac{1}{f(2n+1)}} > \frac{1}{2+\frac{1}{\sqrt{2}-1}} = \sqrt{2}-1$

$\therefore$ $f(1), f(3), f(5)$ has lower limit $\sqrt{2}-1$

using same method we can show $f(2n) - f(2n+2) > 0$ and $\lim_{n\to\infty} f(2n) = \sqrt{2}-1$

$\therefore$ the two sequence of alternative terms are monotonic and they the limit is both $\sqrt{2}-1$, thus the origin sequence is convergent and the limit is $\sqrt{2}-1$