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ANALYSIS I

HOMEWORK #4

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15) Given: E is a metric space
$S \subset E$
$p \in S$ is an interior point
Interior point: $\{p \in S$: there exists a $r > 0$ such that $B(p,r) \subset S\}$

Claim: The set of all interior points of S is an open subset of E that contains all other open subsets of E that are contained in S.

Proof: Let A be an open subset of E that is contained in S and $B(p,r)$ be an open ball in E of radius r.
$A \subset S$ $\quad B(p,r) = \{q \in E : d(p,q) < r\}$.

Consider S^i to be the set of all interior points in S.
$S^i = \{p \in S$: there exists an $r > 0$ s.t. $B(p,r) \subset S\}$

Since A is open, $A = \{p \in A$: there exists an $r > 0$ s.t. $B(p,r) \subset A\}$.
Since $B(p,r) \subset A$ and $A \subset S$, it follows that $B(p,r) \subset A \subset S$ and $B(p,r) \subset S$. It also follows that $A \subset S^i$.

Take $q, s \in B(p,r)$ and let $r = \frac{\varepsilon}{2}$, $\varepsilon > 0$.

Take $q, s \in B(p,r)$ and let $r = \frac{\varepsilon}{2}, \varepsilon > 0$.
For $p \in S^i$, let $B(p,r) \subset S$ where $r = \varepsilon, \varepsilon > 0$.
$\Rightarrow d(q,r) \leq d(q,s) + d(s,r) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
Since $B(q,r) \subset B(p,r) \subset S = B(p, \frac{\varepsilon}{2}) \subset$
$B(p,\varepsilon) \subset S = B(p,r) \subset A \subset S = S^i \subset E$
$q \in S^i$ and S^i is open.

$\Rightarrow$ The set of interior points of S is an open subset of E that contains all other open subsets of E that are contained in S.

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16) a. Given: E - metric Space $S \subset E$.
$\bar{S}$ - the closure of S
Closure of S - the intersection of all closed subsets of E that contain S.

Claim: $\bar{S} \supset S$, S is closed iff $\bar{S} = S$.

Proof: Let $\bar{S} = S_1 \cap S_2 \cap \ldots \cap S_n$. By definition of the closure of S, $S_1 \cap S_2 \cap \ldots \cap S_n$ are closed subsets of E that contain S => $S_1 \cap S_2 \cap \ldots \cap S_n \supset S$. Therefore, $\bar{S} \supset S$.

Assume S is closed and $S \supset S_1 \cap S_2 \cap \ldots \cap S_n$, then since by defin of closure that $S_1 \cap S_2 \cap \ldots \cap S_n$ is closed that S is also closed and $S \supset \bar{S}$.

Assume S is closed and $S \subset S_1 \cap S_2 \cap \ldots \cap S_n$, then since by definition of closure that $S_1 \cap S_2 \cap \ldots \cap S_n$ is closed, then S is also closed and $\bar{S} \supset S$.

Since S is closed by $\bar{S} \supset S$ and $\bar{S} \subset S$, then S is closed iff $\bar{S} = S$.

Note: if S were open, then $S_1 \cap S_2 \cap \ldots \cap S_n$ would also be open & by the definition of closure, $\bar{S} \neq S_1 \cap S_2 \cap \ldots \cap S_n$ if for every $i = 1, 2, \ldots, n$, S_i was an open set

$i = 1, 2, \ldots, n$, S_i was an open set and $S \subset S_1 \cap S_2 \cap \ldots \cap S_n$. $\Rightarrow \bar{S}$ would not be the closure of S.

$\Rightarrow \bar{S} \supset S$ and S is closed iff $\bar{S} = S$.

b) Given: E - metric space, $S \subset E$
$\bar{S}$ - closure of S
$\bar{S} \supset S$ and S is closed iff $\bar{S} = S$

Claim: $\bar{S}$ is the set of all limits of sequences of points of S that converge in E.

Proof: Let p_n be a convergent sequence and $p_n \to p$. Assume $p_n \in S$, then by $S \subset \bar{S}$, $p_n \in \bar{S}$. Since $\bar{S}$ is closed, then $p \in \bar{S}$.

Assume $p \in \bar{S}$, then there needs to exist a $p_n \in S$ s.t. $p_n \to p$. Let $\varepsilon > 0$, $\exists$ an integer N s.t. $|p_n - p| < \varepsilon$, pick an integer M s.t. $|p_m - p| < \varepsilon$ for every $n > N$ and $N < M < N+1$ and $p_m \neq p$. Then, $p_m \in B(p, \varepsilon) \cap S \Rightarrow B(p, \varepsilon) \cap S \neq 0$.
Let $p_n \in B(p, \varepsilon) \cap S$ and $\varepsilon = \frac{1}{n}$. Then, $|p_n - p| < \frac{1}{n} \Rightarrow p_n \in S$ & $p_n \to p$.

c) Given: E metric space. $S \subset E$

$\bar{S}$ - closure of S

$\bar{S} = \cap S_i$, for every S_i, $i = 1, 2, \ldots, n$, S_i is closed

$\bar{S} \supset S$

S is closed iff $\bar{S} = S$

$\bar{S}$ is the set of all limit points of sequences of S that converge in E

Claim: A point $p \in E$, $p \in \bar{S}$ iff any ball in E with center p contains points of S. This is true iff p is not an interior point of S^c.

Proof: This is the same as saying that $\bar{S} = \{p \in E : d(p, S) = 0\}$. For this to be true, then $\forall\, \varepsilon > 0$, $B(p, \varepsilon) \cap S \neq \emptyset$ and $B(p, \varepsilon) \cap S^c \neq \emptyset\}$. $B(p, \varepsilon) \cap S \neq \emptyset$ is true by part (b) when $\varepsilon = 1/n$ and $p_n \in B(p, \frac{1}{n}) \cap S \Rightarrow p \in S$.

For $B(p, \varepsilon) \cap S^c \neq \emptyset$, let p be an interior point of S^c. By definition of an interior point, $B(p, \varepsilon) \subset S^c$ which contradicts $B(p, \varepsilon) \cap S \neq \emptyset \rightsquigarrow$ therefore, $p \notin$ the interior of S^c. Let p be a boundary point of S^c, then $B(p, r) \cap S^c \neq \emptyset$ and $B(p, r) \cap S \neq \emptyset$ as $B(p, r)$ would contain at least one point of S.

$\Rightarrow$ A point $p \in E$ is in $\bar{S}$ iff a ball in E of center p contains points of S. iff

p is not an interior point of S^c.

(18) If $a_1, a_2, a_3, \ldots$ is a bounded sequence of real numbers

$$\lim_{n\to\infty} \sup a_n = \overline{\lim_{n\to\infty}} a_n = \text{l.u.b} \{ x \in \mathbb{R} : a_n > x, \infty \text{ many } n\text{'s} \}$$

$$\lim_{n\to\infty} \inf a_n = \underline{\lim_{n\to\infty}} a_n = \text{g.l.b} \{ x \in \mathbb{R} : a_n < x, \infty \text{ many } n\text{'s} \}$$

a) Prove that $\underline{\lim_{n\to\infty}} a_n \leq \overline{\lim_{n\to\infty}} a_n$

Set $\underline{\lim_{n\to\infty}} a_n = \underline{a}$ and $\overline{\lim_{n\to\infty}} a_n = \bar{a}$. We want to show that $\underline{a} \leq \bar{a}$

Consider $\bar{a} + \varepsilon$. There are only finitely many n's with $a_n \geq \bar{a} + \varepsilon$.

Therefore, all but finitely many a_n's satisfy that $a_n < \bar{a} + \varepsilon$.

Now consider $\underline{a} - \varepsilon$. There are only finitely many n's with $a_n \leq \underline{a} - \varepsilon$.

So, all but finitely many a_n's satisfy that $a_n > \underline{a} - \varepsilon$.

It implies there exist at least one a_n with $\underline{a} - \varepsilon < a_n < \bar{a} + \varepsilon$

$\Rightarrow \underline{a} - \varepsilon < \bar{a} + \varepsilon$

$\Rightarrow \underline{a} < \bar{a} + 2\varepsilon$

$\Rightarrow \underline{a} \leq \bar{a}$ since $\varepsilon > 0$ and arbitrary.

b) $\overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n$ if and only if a_n converges.

($\Rightarrow$) If $\overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n = a$, then a_n converges because all but finitely many a_n's satisfy $a - \varepsilon < a_n < a + \varepsilon$,

where $\varepsilon > 0$ and arbitrarily small.

($\Leftarrow$) $\varepsilon > 0$. There exists N such that $a - \varepsilon < a_n < a + \varepsilon$ for all $n > N$, where

$a - \varepsilon < a_n < \bar{a} + \varepsilon$ for all but finitely many n's. —— ①

$\underline{a} - \varepsilon < a_n < a + \varepsilon$ for all but finitely many n's. —— ②

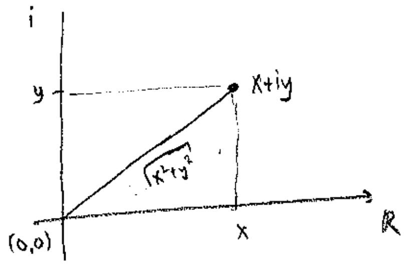
There exist ∞ many a_n's with $a_n > \bar{a} - \varepsilon$ because $\bar{a}$ is the least upper bound of the set where all the elements of that set is less than a_n. From ②, $a + \varepsilon > a_n > \bar{a} - \varepsilon \Rightarrow a \geq \bar{a}$, $\varepsilon > 0$ arbitrarily small.

There exist ∞ many a_n's with $a_n < \underline{a} + \varepsilon$ because $\underline{a}$ is the greatest lower of the set where all the elements of that set is greater than a_n. From ①, $a - \varepsilon < a_n < \underline{a} + \varepsilon \Rightarrow a \leq \underline{a}$, ε arbitrarily small,

$\Rightarrow a \geq \bar{a} \geq \underline{a} \geq a \Rightarrow \bar{a} = \underline{a} = a$

(20) $|z| = d(z, 0)$

a) $|x+iy| = d(x+iy, 0)$, where x and $y \in \mathbb{R}$.



$$|x+iy| = d((x,y),(0,0)) = \sqrt{x^2+y^2}$$

b) Suppose that $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = x_1 + x_2 + i(y_1+y_2)$

$\Rightarrow |z_1 + z_2| = |x_1 + x_2 + i(y_1+y_2)|$

It is known that $|z_1| = \sqrt{x_1^2+y_1^2}$ and $|z_2| = \sqrt{x_2^2+y_2^2}$, (from the definition of $|x+iy| = \sqrt{x^2+y^2}$ of part a)

We need to show $|z_1+z_2| = |x_1+x_2+i(y_1+y_2)| = \sqrt{(x_1+x_2)^2+(y_1+y_2)^2} \le \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}$

Corollary at page 35 from Rosenlicht tells that

$$\sqrt{(a_1+b_1)^2+(a_2+b_2)^2+\dots+(a_n+b_n)^2} \le \sqrt{a_1^2+a_2^2+\dots+a_n^2} + \sqrt{b_1^2+b_2^2+\dots+b_n^2}$$

This applies to

$$\sqrt{(x_1+x_2)^2+(y_1+y_2^2)} \le \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}$$

Therefore, $|z_1+z_2| \le |z_1| + |z_2|$

c) Suppose $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then, $|z_1 z_2| = |(x_1+iy_1)(x_2+iy_2)| = |x_1x_2 + ix_1y_2 + iy_1x_2 + \underbrace{i^2}_{-1} y_1y_2|$

$\Rightarrow |z_1z_2| = |x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2)|$

$= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2}$

It is known that $|z_1| = \sqrt{x_1^2+y_1^2}$ and $|z_2| = \sqrt{x_2^2+y_2^2}$

$|z_1z_2| = \sqrt{(x_1x_2-y_1y_2)^2+(x_1y_2+y_1x_2)^2} = \sqrt{x_1^2x_2^2 - \cancel{2x_1x_2y_1y_2} + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2 + \cancel{2x_1y_2x_2y_1}}$

$= \sqrt{x_1^2x_2^2 + x_1^2y_2^2 + y_1^2y_2^2 + y_1^2x_2^2} = \sqrt{x_1^2(x_2^2+y_2^2) + y_1^2(x_2^2+y_2^2)} = \sqrt{(x_2^2+y_2^2)(x_1^2+y_1^2)}$

$= \sqrt{x_1^2+y_1^2} \cdot \sqrt{x_2^2+y_2^2} = |z_1| \cdot |z_2|$.

24) S complete subspace of E
Since S is complete all Cauchy sequences must converge, in S. So any convergent series is Cauchy and therefore must converge in S. So ~~[illegible]~~ S must be closed.

26) $S = \{\frac{1}{n} + \frac{1}{m} : n, m \in \mathbb{Z}^+\}$ Any cluster point in S has at least one sequence that converges to it. Any convergent series in S is of the form

$a_n = \frac{1}{n} + \frac{1}{m}$, $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} + \frac{1}{m} = \frac{1}{m}$.

So the set of cluster points is
$\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{m}\}$ $m \in \mathbb{Z}^+$. Taking the limit as m goes to infinity yields 0 as a cluster point so our set of cluster points is:
$\{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\}$

28 $S \subset E$, E metric space.
Suppose S contains all of its cluster points. Take any convergent series in S. $p_n \to p$. The ~~point~~ limit of the convergent series p is itself a cluster point since any $B(p, \epsilon)$ $\epsilon > 0$ must contain infinitely many points since p_n is convergent. So every convergent sequence in S converges to a point in S, hence S is closed.
Suppose S is closed, let p be a cluster point of S, by way of contradiction suppose $p \notin S$. Since p is a cluster point any open ball around p contains infinitely many points. We can construct a sequence $p_n \in B(p, \frac{1}{n})$ so for any $\epsilon > 0$ pick N such that $\epsilon = \frac{1}{N}$ so for $n > N$ $d(p_n, p) < \epsilon$ so $\{p_n\}$ must converge. Since $p \notin S$ S must be open, a contradiction.

30) a) Infinite subset of $\mathbb{R}$ with no cluster point : $\mathbb{Z}$

The set of integers is an infinite subset of $\mathbb{R}$

To have a cluster point we need to find a point p s.t. an open ball centered at p contains infinitely many points.

However, for any finite radius ball we have finitely many integers contained in the ball.

Therefore, set of integers has no cluster points.

b) A complete metric space, that is bounded but not compact : $E=(0,1)$

$$d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{o/w.} \end{cases}$$

The metric space is bounded but not compact. If we take the open cover $E \subset \bigcup_{x\in(0,1)} B(x,r)$ $r<1$, there is no finite subcover.

Only Cauchy Sequences in the metric space are in the form $\{a, a, \dots\}$ and obviously they converge to $a \in E$. Therefore the metric space is complete.

c) A metric space none of its closed balls are complete: $\mathbb{Q}$, $d(x,y)=|x-y|$

Take any closed ball in the metric space: $\overline{B(x,r)} = \{y \in \mathbb{Q} : |x-y| \le r\}$

$\bar{y} = x + \frac{\sqrt{2}}{N}$ is irrational for any N, and in the closed ball for large enough N for each r.

Therefore we can take a Cauchy sequence in a closed ball of the metric space which does not converge in the metric space

32) Union of finite # of subsets of a metric space is compact.

E is a metric space

Take a collection of compact subsets of E : $S_i \subset E$, S_i compact $i=1,\dots,n$

Let $S = \bigcup_{i=1}^{n} S_i$, and let $\bigcup_{j=1}^{\infty} U_j$ be an open cover of S.

For each $i=1,\dots,n$

Since we have $S_i \subset S \subset \bigcup_{j=1}^{\infty} U_j$ and S_i is compact, there must be a finite subcover :

$$S_i \subset \bigcup_{j \in J_i} U_j \quad J_i \text{ is a finite set.}$$

Now consider the union $\bigcup_{i=1}^{n} J_i$, since n is finite and J_i are finite sets, this set is finite. Call this union K.

Then $\bigcup_{j \in K} U_j$ is a finite subcover containing $S = \bigcup_{i=1}^{n} S_i$

Therefore S is compact.

33) E is a compact metric space, $\{U_i\}_{i \in I}$ a collection of open subsets of E s.t $\bigcup_{i \in I} U_i = E$. We want to show $\exists\, \varepsilon > 0$ s.t any closed ball in E with radius ε is entirely contained in at least one U_i.

Suppose the claim is not true: For each $\varepsilon > 0$, there exists a closed ball with radius ε which is not contained in any of U_i's.

Take $\varepsilon = 1/n$, due to our assumption for each n there is at least one closed ball $\overline{B(x_n, 1/n)}$ which is not contained in any of U_i's.

Note that sequence of centers of balls, $\{x_n\}$, has a convergent subsequence in E, since E is compact. Call that limit x. $(x \in E)$

Then we know $x \in U_i$ for some i. (we'll call that set just U)

Take an open ball around x in U : $B(x, r) \subset U$, $r > 0$

Take $\bar{n}$ s.t $\frac{1}{\bar{n}} < \frac{r}{2}$, $d(x_{\bar{n}}, x) < r/2$

Then we get $B(x_{\bar{n}}, \frac{1}{\bar{n}}) \subset B(x, r) \subset U$ $\times$ Contradiction