Homework #5

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$\boxed{17.}$ boundary of $S = \overline{S} \cap \overline{S^c}$

(a) Show that E is the disjoint union of the interior of S, the interior of S^c, and the boundary of S.

E is the disjoint union of S and S^c: $E = S \cup S^c$

Suppose S is closed, then S^c is open.

S^c is open $\Rightarrow$ any $p \in S^c$ is the center of some open ball that is entirely contained in S^c.

$\Rightarrow$ any $p \in S^c$ is an interior point of S^c.

$\Rightarrow$ S^c is equal to the interior of S^c.

S is closed $\Rightarrow$ for a point $p \in S$ one and only one of the following two statements is true:

- there exists some open ball with center p that is entirely contained in S
  $\Rightarrow$ p is in the interior of S.
- any open ball with center p contains points of S and points of S^c.
  $\Rightarrow$ p is in $\overline{S}$ and in $\overline{S^c}$: $p \in \overline{S} \cap \overline{S^c}$
  $\Rightarrow$ p is a point of the boundary of S.

$\Rightarrow$ an arbitrary point of E is either in the interior of S, or in the boundary of S, or in the interior of S^c.

$\Rightarrow$ $E =$ (interior of S) $\cup$ (boundary of S) $\cup$ (interior of S^c), which is a disjoint union.

(It was assumed that S is closed. Likewise the result can be shown for the case that S is open, by simply exchanging S and S^c in the statements above.)

(b) Show that S is closed if and only if S contains its boundary.

"if": S contains its boundary

$\Rightarrow$ there exist $p \in S$ with $p \in \overline{S} \cap \overline{S^c}$ $\Rightarrow$ $p \in \overline{S^c}$

Since $p \in \overline{S^c}$, any open ball with center p contains points of S^c

$\Rightarrow$ Since $p \in S$, S is closed.

"only if": S is closed.

$\Rightarrow$ there exist $p \in S$ such that any open ball with center p contains points of S and points of S^c.

$\Rightarrow$ p is in $\overline{S}$ and in $\overline{S^c}$: $p \in \overline{S} \cap \overline{S^c}$

$\Rightarrow$ Since $p \in S$, S contains its boundary.

(c) Show that S is open if and only if S and its boundary are disjoint.

"if": S and its boundary are disjoint.

$\Rightarrow$ any point $p \in S$ is not in $\overline{S} \cap \overline{S^c}$, hence $p \in S$ is not in $\overline{S^c}$

$\Rightarrow$ not every ball with center p contains points of S^c, thus there is some open ball with center p that is entirely contained in S.

$\Rightarrow$ Since p is any point in S, S is open.

"only if": S is open.

$\Rightarrow$ for any point $p \in S$ there is an open ball with center p such that the ball is entirely contained in S. Hence, this ball does not contain points of S^c.

$\Rightarrow p \notin \overline{S^c} \Rightarrow p \notin \overline{S} \cap \overline{S^c}$

$\Rightarrow$ Since p is any point in S, S does not contain its boundary.
Hence, S and its boundary are disjoint.

19. Let $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ be bounded sequences of real numbers. Show

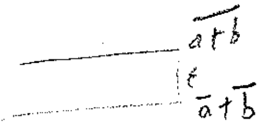
$$\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

w/ equality if $a_n \to a$. (or $b_n \to b$)

If

Suppose! $\underbrace{\limsup_{n\to\infty} a_n}_{\overline{a}} + \underbrace{\limsup_{n\to\infty} b_n}_{\overline{b}} < \underbrace{\limsup_{n\to\infty}(a_n+b_n)}_{\overline{a+b}}$ (*)

Then let $d(\overline{a+b}, \overline{a}+\overline{b}) = \overline{a+b} - (\overline{a}+\overline{b}) = \epsilon$

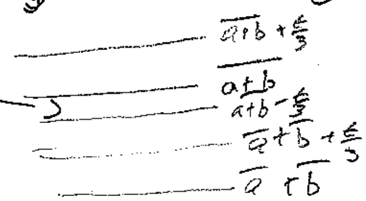


Now, we know that $\exists$ infinite n s.t $(a_n+b_n) > \overline{a+b} - \frac{\epsilon}{3}$ b/c $\overline{a+b}$ is the limsup. Furthermore we know, b/c $\overline{a+b}$ is the $\limsup_{i\to\infty}(a_i+b_i)$ that there are only finite $(a_n+b_n) > \overline{a+b} + \frac{\epsilon}{3}$.

Now consider (*)

if $\overline{a} + \overline{b} < \overline{a+b}$, then $\overline{a} + \frac{\epsilon}{6} + \overline{b} + \frac{\epsilon}{6}$

$= \overline{a} + \overline{b} + \frac{\epsilon}{3} < \overline{a+b} + \frac{\epsilon}{3}$.



Since we had an infinite # of (a_i+b_i) s.t $\overline{a+b} - \frac{\epsilon}{3} < a_i + b_i < \overline{a+b}$, there must be an infinite # of (a_i+b_i)'s s.t

$a_i + b_i > \overline{a}+\overline{b} + \frac{\epsilon}{3}$ (since $\overline{a+b} - (\overline{a}+\overline{b}) = \epsilon$)

However, since $\overline{a}$ is the limsup of a_i, $\exists$ only finite # of $a_i > a_i + \frac{\epsilon}{6}$. Likewise for the b_i. This implies only a finite # of $(a_i+b_i) > \overline{a}+\overline{b}+\frac{\epsilon}{3}$.

(In fact the # would be @ most $2[\#a_i > a_i + \frac{\epsilon}{6}] + 2[\#b_i > b_i + \frac{\epsilon}{6}]$.

This is a contradiction. $\longrightarrow$

Now, we must show equality when $a_n \to a$ or $b_n \to b$.
WLOG. just assume $a_n \to a$.

Now suppose

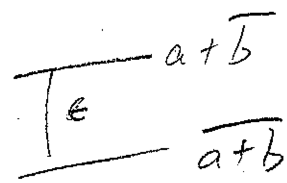
$$* \quad \limsup_{n\to\infty}(a_n+b_n) < \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty}(b_n)$$

Since $a_n \to a$, we have $\limsup_{n\to\infty}(a_n) = \lim_{n\to\infty}(a_n) = a$

So,
Show $(**)$ $\limsup_{n\to\infty}(a_n+b_n) < a + \limsup_{n\to\infty}(b_n)$

or

$$\overline{a+b} < a + \overline{b} \quad \Rightarrow$$



As before let $d(\overline{a+b}, a+\overline{b}) = \epsilon$.

Now, since $\overline{a+b}$ is the $\limsup(a_i+b_i)$
we can have only finitely many
a_i+b_i s.t. $a_i+b_i > \overline{a+b} + \frac{\epsilon}{8}$

However since a converges, $\underset{a=\overline{a}}{\text{we}}$ can pick N large enough s.t.
$$d(a, a_n) < \frac{\epsilon}{2}, \quad \text{for } n > N$$
Furthermore we can find infinitely many b_i's s.t. $\overline{b} - \frac{\epsilon}{4} < b_i < \overline{b} + \frac{\epsilon}{4}$, since $\overline{b}$ is the limsup.
Hence $\forall i > N$, we have $a+\overline{b} - \frac{3\epsilon}{4} < a_i+b_i < a+\overline{b} + \frac{3\epsilon}{4}$. But since,
$$a+\overline{b} - \frac{3\epsilon}{4} > \overline{a+b} + \frac{\epsilon}{8},$$ this is a contradiction.

21) Let $z_1, z_2 \ldots$ and $w_1, w_2 \ldots$ be convergent sequences of complex numbers with $\lim_{n\to\infty} z_n = z$, $\lim_{n\to\infty} w_n = w$, which are bounded.

Consider $\| z_n + w_n - z - w \| = \| z_n - z + w_n - w \|$

$$\leq \| z_n - z \| + \| w_n - w \| = \varepsilon/2 + \varepsilon/2 \text{ for sufficiently large } n.$$

So $\| z_n + w_n - z - w \| \leq \varepsilon$. for sufficiently large n.

Thus, $\lim_{n\to\infty} (z_n + w_n) = z + w$

$\lim (z_n - w_n) = z - w$ clearly by considering $\| z_n - w_n - (z - w) \|$

$$\leq \| z_n - z \| + \| w - w_n \|$$
$$\leq \varepsilon.$$

Now consider ~~Rates~~ $\| zw - z_n w_n \| = \| zw + z_n w - z_n w - z_n w_n \|$

$$= \| w(z - z_n) + z_n (w - w_n) \|$$

Because $z_1, z_2 \ldots$ and $w_1, w_2 \ldots$ are bounded, $\exists M \in \mathbb{R}$ with $|z_n| < M$ and $|w_n| < M$

So $\| w(z - z_n) + z_n (w - w_n) \|$

$$\leq \| w(z - z_n) \| + \| z_n (w - w_n) \| < \| w(z - z_n) \| + \| M (w - w_n) \|$$

Now, for sufficiently large n: $\leq (w)\left(\frac{\varepsilon}{2w}\right) + M\left(\frac{\varepsilon}{2M}\right)$

$$\leq \varepsilon$$

So $\lim_{n\to\infty} (z_n w_n) = zw$.

$\lim_{n\to\infty} \left(\frac{z_n}{w_n}\right) = \frac{z}{w}$ for $w_n, w \neq 0$ can be shown similarly by considering $\| \frac{z}{w} - \frac{z_n}{w} + \frac{z_n}{w} - \frac{z_n}{w_n} \| = \| \frac{1}{w}(z - z_n) + z_n \left(\frac{1}{w} - \frac{1}{w_n}\right) \|$ and proceeding similarly since $\frac{1}{w} - \frac{1}{w_n} < \varepsilon$ for sufficiently large n.

27. nonempty subset S of $\mathbb{R}$ that is bounded from above and has no greatest element.
Prove that l.u.b S is a cluster point of S.

Since S is nonempty and bounded from above, $l = \text{l.u.b.} S$ exists.
S has no greatest element, thus l is not an element of S.
$\Rightarrow$ for any $p \in S$: $p < l$
for any $\varepsilon > 0$, $l - \varepsilon, l - \frac{\varepsilon}{2}, l - \frac{\varepsilon}{4} \dots$ are in S.
$\Rightarrow$ there exist $a \in S$ with $|l - a| < \varepsilon$, $b \in S$ with $|l - b| < \frac{\varepsilon}{2}$, ...
this can be continued infinitely, there are ∞ many points in S that are arbitrarily close to l.
$\Rightarrow$ for each $\varepsilon > 0$ there are infinitely many points $p \in S$ such that $|l - p| < \varepsilon$.
$\Rightarrow$ any ball with center l and radius $\varepsilon > 0$ contains infinitely many points of S.
$\Rightarrow$ l.u.b. S is a cluster point of S.

29. Let S be a subset of a M.S. E and let $p \in E$. Show p is a cluster pt of S $\Leftrightarrow$ p is the limit of a Cauchy sequence in $S \cap \{p\}^c$.

$(\Rightarrow)$ p is a cluster point of S. Show p is limit of a Cauchy sequence in $S \cap \{p\}^c$.

Consider $B(p, \epsilon) \subset$ infinite pts in S for any $\epsilon > 0$.

Pick $p_1 \in B_1(p, 1)$, $p_2 \in B_2(p, \frac{1}{2})$, ...

We must ensure $p_1 \neq p$, $p_2 \neq p$, etc. But since each B_n contains an infinite # of pts which are in S, each B_n contains an infinite # of pts in S which are not p. So never pick p & we have a Cauchy Sequence of pts in $S \cap \{p\}^c$ which converge to p.

$\Leftarrow$ p is the limit of a Cauchy Sequence in $S \cap \{p\}^c$.

Then for every $\epsilon > 0$ we have $d(p, p_n) < \epsilon$ for some integer N and $n > N$. Hence any $B(p, \epsilon)$ contains an infinite number of pts p_j w/ the $p_j \in S \cap \{p\}^c$, $S \supset S \cap \{p\}^c$, all of these $p_j \subset S$. So, p is a cluster point of S.

31. Let $a, b \in \mathbb{R}$ $a < b$. Prove $[a,b]$ compact.

Let $\{U_i\}_{i \in I}$ be a collection of open subsets of $\mathbb{R}$ whose union contains $[a,b]$. Let $S = \{ x \in [a,b] : x > a$ and $[a, x]$ is contained in the union of $\{U_j\}_{j \in J}$, $J \subset I$, J is finite$\}$.

S must be non-empty, b/c $a \in U_i$ for some i. This means we have some $\epsilon > 0$ s.t. $B(a, \epsilon) \subset U_i$. Hence $\exists$ $x \in B(a, \epsilon)$ s.t. $x > a$ and $x \in U_i$ as well.

Since $x \in [a,b]$ if $x \in S$, then l.u.b. $S \leq b$. Hence l.u.b $S \in [a,b]$ & so lub $S \in U_i$ for some i.

Suppose, now that $y = \text{lub} S$ and $y < b$. Then we know $y \in U_i$ for some i, so $\exists$ $\epsilon > 0$ so that $B(y, \epsilon) \subset U_i$. Hence $(y+\epsilon) \in U_i$ and $[a, y+\epsilon]$ is also contained in a finite union of U_i. Hence y was not a least upper bound.

(Note: There need be no fear of jumping over b here b/c if $y + \epsilon \in U_i$ so is $y + \frac{\epsilon}{n}$ for any $n \geq 1$ so we can hav $d(y, y+\frac{\epsilon}{n}) < d(y, b)$)

34). Suppose $S \times T$ is compact, but S is not compact. So S is not both closed and bounded

- If S is not closed, $\exists x \in S'$ with $x_0 \notin S$ but $x_0 \in B(x, \epsilon)$. $\forall \epsilon$
  Now examine $z \in S \times T$ with x as the first n coordinates. That is, if $x = (x_1, x_2 \ldots x_n)$, $z = (x_1, x_2, \ldots, x_n, y_1, \ldots, y_m)$. And z_0 be a point in $S \times T$ with x_0 as the first n coordinates. Since $x_0 \notin S$, $z_0 \notin S \times T$.
  But $B(z, \epsilon)^{\forall \epsilon}$ contains z_0, since $d(z, z_0) = d(x, x_0) < \epsilon$. This implies that $S \times T$ is not closed. Contradiction, since $S \times T$ is compact
  Also note, this proves that if defined the same way, if x is a boundary point of S, then $z = (x_1, x_2 \ldots x_n, y_1, \ldots, y_m)$ is a boundary point of $S \times T$. This will be used later.

- If S is not bounded, $\sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \nleq M$ for any M for an $x \in S$. Observe $z \in S \times T$
  $z = (x_1, x_2, \ldots, x_n, y_1 \ldots y_n)$. Since $y_1^2, \ldots, y_m^2$ are nonnegative, $\sqrt{x_1^2 + x_2^2 + \ldots + x_n^2 + y_1^2 + \ldots + y_m^2} \nleq M$ for any M as well. This implies $S \times T$ is not bounded, but it's compact. This contradicts the assumption.

Now, if $S \times T$ is compact, we've shown S and T must be bounded and closed. They are both subsets of E^a for some a, so S, T are compact.

Finally assume $S \times T$ is open, but S is not open. This means that S contains one of its boundary points, called $x \in S$. We showed earlier that if x is a boundary point of S, then $z = (x_1, \ldots, x_n, y_1 \ldots y_m)$ is a boundary point of $S \times T$. But $S \times T$ is open, so it cannot contain any boundary points; yet $z \in S \times T$ since $x \in S$. This contradicts that S is not open.

So far we've shown:

$S \times T$ open $\longrightarrow$ S, T open
$S \times T$ closed $\longrightarrow$ S, T closed
$S \times T$ bounded $\longrightarrow$ S, T bounded
$S \times T$ compact $\longrightarrow$ S, T compact

Lemma

If $S \subset E^n$, $T \subset E^m$, $S \times T \subset E^{n+m}$.

$x \in S \Rightarrow x = (x_1, x_2, \ldots x_n)$

$y \in T \Rightarrow y = (y_1, \ldots y_m)$

$z \in S \times T \Rightarrow z = (x_1, \ldots x_n, y_1, \ldots y_m)$

Then z is a boundary point of $S \times T \Leftrightarrow$ x is a boundary point of S or y is a boundary point of T.

pf)

($\Leftarrow$) WLOG x_1 is a boundary pt of S. Hence $\forall \epsilon > 0$ $B(x_1, \epsilon) \supset$ pts of S & S^c. Let $x_0 \in S^c$ and $x_0 \in B(x_1, \epsilon)$ for some $\epsilon > 0$. Now $B(x_1, \epsilon) \in E^n$ so $(B(x_1, \epsilon) \times T) \in E^{n+m}$. The set $B(x_1, \epsilon) \times T$ clearly contains the pts

$$\left.\begin{array}{l} z_1 = (x_{11}, \ldots x_{1n}, y_{01}, \ldots y_{0m}) \\ z_0 = (x_{01}, \ldots x_{0n}, y_{01}, \ldots y_{0m}) \end{array}\right\}$$

for some $y_0 \in T$.

$$d(z_1, z_0) = \sqrt{(x_{11} - x_{01})^2 + \cdots (x_{1n} - x_{0n})^2 + 0} < \epsilon$$

Now since $x_1 \in S$ & $y_0 \in T \Rightarrow z_1 \in S \times T$
$z_0 \in S^c \times T$

Hence for $x_1 \in S$ a boundary pt & any $\epsilon > 0$ we can always find $x_0 \in B(x_1, \epsilon)$ s.t. $d(z_1, z_0) < \epsilon$ w/ $z_1 \in S \times T$ & $z_0 \in S^c \times T$, so z_1 is a boundary point.

back $\rightarrow$

($\Rightarrow$) z_1 is a boundary point of $S \times T$. Suppose $z_1 = (x_1, y_1)$ w/ neither x_1 nor y_1 a boundary point of S or T, respectively.

This $\Rightarrow$ that $\exists\, \epsilon > 0$ s.t. $B(x_1, \epsilon) \subset S$ and $\exists\, \delta > 0$ s.t. $B(y_1, \delta) \subset T$.

WLOG assume $\delta = \min(\epsilon, \delta)$.

Consider $z_1 = (x_1, y_1)$. Take $B(z_1, \delta)$
$$\Rightarrow \sqrt{(x_1 - x_{11})^2 + \cdots \quad (y_m - y_{1m})^2} < \delta$$
coordinates points ... (you get it right?)

Certainly this means
$$\sum_{i=1}^{n} (x_1 - x_{1i})^2 < \delta^2$$
and
$$\sum_{i=1}^{m} (y_1 - y_{1i})^2 < \delta^2$$
Hence all such $z \in B(z_1, \delta)$ have $x \in S$ & $y \in T$ so that z_1 is <u>not</u> a boundary point, a contradiction

Now assume S, T are compact, but $S \times T$ is not compact. So $S \times T$ is either not closed or not bounded.

If $S \times T$ is not closed, it $\exists z \in S \times T$ with $z_0 \notin S \times T$ but $z_0 \in B(z, \varepsilon)$. So $d(z, z_0) < \varepsilon$. Take $x \in S$ to be the first n coordinates of z, and $x_0 \notin S$ to be the first n coordinates of z_0. Surely $d(z, z_0) \geq d(x, x_0)$. So $d(x, x_0) < \varepsilon$. So, $x_0 \in B(x, \varepsilon)$. But this implies S is not closed, a contradiction.

If $S \times T$ is not bounded, $\sqrt{x_1^2 + x_2^2 + \ldots + x_n^2 + y_1^2 + \ldots + y_m^2} \nleq M$ for any M,

So $x_1^2 + \ldots + x_n^2 + y_1^2 + \ldots + y_n^2 \nleq M^2$ for any M. Since T is bounded,

$$x_1^2 + \ldots + x_n^2 \nleq M^2 - (y_1^2 + \ldots + y_m^2)$$

Then call $M^2 - (y_1^2 + \ldots + y_m^2) = A$.

So $\sqrt{x_1^2 + \ldots + x_n^2} \nleq A$ for any A. So S is not bounded. This is a contradiction since S is compact.

So now, S, T are compact $\Longrightarrow$ $S \times T$ is closed, bounded subset of E^{m+n}

So $S \times T$ is compact.

Finally, suppose S, T are open, but $S \times T$ is not open. We've previously proven that if x is a boundary point of S, then $z = (x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a boundary point of $S \times T$. Since $S \times T$ is not open, it contains a boundary point.

Given that $z = (x_1, \ldots, x_n, y_1, \ldots, y_m)$ is the boundary point, either x or y must a be boundary point of S or T respectively. But since $z \in S \times T$, $x \in S$, $y \in T$. But since one is a boundary point of an open set (recall S & T are open) they cannot both be in their respective sets. This contradicts that $S \times T$ is not open.

So we've now proven:

$S \times T$ open $\Longleftrightarrow$ S, T open

$S \times T$ closed $\Longleftrightarrow$ S, T closed

$S \times T$ bounded $\Longleftrightarrow$ S, T bounded

$S \times T$ compact $\Longleftrightarrow$ S, T compact

[35.] Prove that a metric space is sequentially compact if and only if every infinite subset has a cluster point.

"if": every infinite subset has a cluster point.

Consider the set of terms $\{p_1, p_2, \dots\}$ of an arbitrary sequence

- If the set $\{p_1, p_2, \dots\}$ is finite, then at least one $p \in \{p_1, p_2, \dots\}$ is repeated infinitely many times in the sequence.

  $\Rightarrow$ $p, p, p, \dots$ is a convergent subsequence of the sequence $p_1, p_2, p_3, \dots$

- If the set $\{p_1, p_2, \dots\}$ is infinite, then the set is an infinite subset of the metric space. Hence, the set $\{p_1, p_2, \dots\}$ has a cluster point p.

  Since any ball with center p contains an infinite number of elements of the set, one can pick a p_{n_1} of the set with $d(p_{n_1}, p) < 1$, then a p_{n_2} with $d(p_{n_2}, p) < \frac{1}{2}, \dots$

  Thus one can construct a subsequence that converges to p.

$\Rightarrow$ every sequence has a convergent subsequence.

$\Rightarrow$ the metric space is sequentially compact.

"only if": a metric space is sequentially compact.

Consider a sequence $p_1, p_2, p_3, \dots$ in the infinite subset S of the metric space.

Since the metric space is sequentially compact, the sequence $p_1, p_2, p_3, \dots$ has a convergent subsequence $p_{n_1}, p_{n_2}, p_{n_3}, \dots$ with limit p.

$\Rightarrow$ for any $\varepsilon > 0$ there exists N such that $|p_{n_n} - p| < \varepsilon$ for all $n_n > N$

$\Rightarrow$ any open ball with center p and some radius $\varepsilon > 0$ contains an infinite number of elements of the subsequence, and thus infinitely many elements of the sequence. Since the sequence is in the infinite subset S, any open ball with center p contains an infinite number of elements of S. Hence, p is a cluster point of S.

$\Rightarrow$ every infinite subset of a sequentially compact metric space has a cluster point.

37). i $\rightarrow$ ii. This is proven in corollary 1 of section 5.

ii $\rightarrow$ iii: If E is sequentially compact, every sequence has a convergent (Cauchy) subsequence. Problem 36 states that if every sequence has a Cauchy subsequence then the space is totally bounded. Since it is totally bounded, it is the union of finitely many closed balls, so it is closed. Thus all Cauchy sequences converge in E, so E is complete.

iii $\rightarrow$ i: Assume E is totally bounded and complete but not compact.

We know that $E = \bigcup_{i \in I} B_i$, a finite union of closed balls.

Also, by assumption, $\exists \{U_i\}_{i \in I} \supseteq E$ s.t. there is no finite subcover.

So $E = (E \cap B_1) \cup (E \cap B_2) \ldots \cup (E \cap B_n)$

The remainder of the proof follows precisely from the final proof of section 5.

an open (closed) subset $S \subseteq E$ is connected iff it is not the disjoint union of two nonempty open (closed) subsets of E

Contrapositive. $\sim B \Rightarrow \sim A$

If S is the disjoint union of two nonempty open (closed) subsets of E $\Rightarrow$ S, open (closed) subset of E is not connected.

If two subsets of E, S_1 and S_2 are open (closed) and nonempty and their union is S, their intersection $\{\emptyset\}$, then $S_1 \cup S_2 = S$ and S_2 must be the compliment of S_1 in S, $S_2 = S_1^c$. $S = S_1 \cup S_1^c$. By the same argument $S = S_2 \cup S_2^c$ $S_1 = S_2^c$. Since S_1, S_2 are both open (closed) then their compliments are closed (open) and since $S_1 = S_2^c$, $S_2 = S_1^c$ then S_1 and S_2 are both open and closed (closed and open)

This means $\exists\ S_1, S_2 \neq \emptyset, S$ in S that are open and closed, and S is not connected.

38 Cont

Converse, once again using contrapositive.

$$\sim A \Rightarrow \sim B$$

An open (closed) subset of

E, S is not connected $\Rightarrow$ $S = S_1 \cup S_2$, S_1, S_2 nonempty

and open (closed).

If is is not connected then $\exists$ a subset of S, S_1,

s.t. S_1 is open and closed, $S_1 \neq \emptyset$ or S.

let $S_2 = S_1^c$ in S, then since S_1 is open and closed, S_2 is open and closed.

Then $S = S_1 \cup S_2$ $S_1 \cap S_2 = \emptyset$ and S_1, S_2 are both nonempty and are both open (closed) since they are both open and closed.

QED.