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HOMEWORK

1) a) We need to show that for any $\varepsilon > 0$, there exists $\delta > 0$ so that $p \in E$ and $d(p, p_0) < \delta$, then $d'(f(p), f(p_0)) < \varepsilon$

Let us check $p_0 = 0$

If $p < 0$, then $d(p, 0) = p$ and $d'(f(p), f(0)) = d'(0, 0) = 0$

If $p \geq 0$, then $d(p, 0) = p$ and $d'(f(p), f(0)) = d'(p, 0) = p$

Choose $\varepsilon > p$ and $\varepsilon = \delta$, for every $\varepsilon > 0$, there exists $d(p, 0) = p < \delta$ and $d'(p, 0) = p < \varepsilon$

$\boxed{1(d)}$ Discuss the continuity of the function $f : \mathbb{R} \to \mathbb{R}$ if f is given by:

$$\Rightarrow \quad f(x) = \begin{cases} 0 & \text{if } x \text{ is not rational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ where } p \text{ and } q \text{ are integers with no common divisors other than } \pm 1, \text{ and } q > 0. \end{cases}$$

Claim: f is not continuous.

Proof: Pick any $x_0 \in \mathbb{R}$, is f continuous at x_0? If it was then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathbb{R}$ and $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$. From (LUB 5.) on page 26, there exists a rational number $\frac{r}{s} \in B(x_0; \delta)$ for any $\delta > 0$. (we may assume r, s have no common factors.) Now pick any irrational numbes $k \in \mathbb{R}$ and choose a positive integer N large enough so that $\frac{r}{s} + \frac{k}{N} \in B(x_0; \delta)$.

$\Rightarrow$ If x_0 is irrational then $|f(x) - f(x_0)| = |f(x)| < \varepsilon$. Then letting $x = \frac{r}{s}$ we see that $|f(x)| = \frac{1}{s} < \varepsilon$, so picking $\varepsilon \leq \frac{1}{s}$ leads to a contradiction.

$\Rightarrow$ If $x_0 = \frac{p}{q}$ then $|f(x) - f(x_0)| = |f(x) - \frac{1}{q}| < \varepsilon$. Now let $x = \frac{r}{s} + \frac{k}{N}$ so that $|f(x) - \frac{1}{q}| = \frac{1}{q} < \varepsilon$ and once again pick $\varepsilon \leq \frac{1}{q}$. Thus, f is not continuous.

$\boxed{2}$ Let E, E' be metric spaces, $f: E \to E'$ a continuous function. Show that if S is a closed subset of E', then $f^{-1}(S)$ is a closed subset of E. Derive from this the results that if f is a continuous real-valued function on E then the sets $\{p \in E : f(p) \leq 0\}$, $\{p \in E : f(p) \geq 0\}$, and $\{p \in E : f(p) = 0\}$ are closed.

<u>Proof</u>: We wish to show that $\mathscr{C}(f^{-1}(S))$ is an open subset of E. Since $S \subset E'$ is closed, $\mathscr{C}(S) \subset E'$ is open and from the proposition on page 70 we know that $f^{-1}(\mathscr{C}(S)) \subset E$ is open. It follows that:

$$\mathscr{C}(f^{-1}(S)) = \{p \in E : f(p) \notin S\}$$

$$= \{p \in E : f(p) \in \mathscr{C}(S)\}$$

$$= f^{-1}(\mathscr{C}(S)).$$

$\Rightarrow$ Then it follows immediately that $f^{-1}(S)$ is closed.

Now since $\{x \in \mathbb{R} : x \leq 0\}$, $\{x \in \mathbb{R} : x \geq 0\}$, $\{0\}$ are all closed subsets of $\mathbb{R}$, it follows that if $f: E \to \mathbb{R}$ then $\{p \in E : f(p) \leq 0\}$, $\{p \in E : f(p) \geq 0\}$, and $\{p \in E : f(p) = 0\}$ are closed subsets of E. ▨

[4] Given $f: U \to V$, strictly increasing and onto, where $U, V \subset \mathbb{R}$ are open or closed intervals.

Claim: f is continuous on U.

Proof: Note that U, V are bounded, f is one-one.

$\Rightarrow$ If U, V are closed then we can see that U, V are nonempty compact subsets of $\mathbb{R}$.

Assume f is not continuous. Then by corollary 2 on pg. 78, f does not attain a maximum at any $u \in U$ or attain a minimum at any $u \in U$.

$\Rightarrow$ Since V is a nonempty compact set $a = \text{lub}\, V$, $b = \text{glb}\, V \in V$. but f is onto so $\exists\, a_0, b_0 \in U$ such that $f(a_0) = a$ and $f(b_0) = b$. Then, $f(b_0) \leq v$, $f(a_0) \geq v$ for all $v \in V$. Once again, since f is onto this means:

$$f(b_0) \leq f(u) \quad \text{for all } u \in U.$$
$$f(a_0) \geq f(u)$$

A contradiction. Thus, f is continuous.

$\Rightarrow$ Now consider the case when U, V are open.

Once again, assume f is not continuous.

(continued $\Rightarrow$)

$\boxed{4}$ continued...

Then by the proposition on page 70, there must exist some open subset $B \subset V$ such that

$$\Rightarrow \qquad f^{-1}(B) = \{u \in U : f(u) \in B\}$$

is not an open subset of U. Now pick $u_0 \in f^{-1}(B)$. Then $f(u_0) \in B$, but since B is open $\exists\, \varepsilon > 0$ such that $|f(u) - f(u_0)| < \varepsilon$ for all $u \in U$ this holds. But now if we pick $\delta > 0$ such that $|u - u_0| < \delta$ then $|f(u) - f(u_0)| < \varepsilon$ where $u \in f^{-1}(B)$.

$\Rightarrow$ But this implies that $f^{-1}(B)$ is open, a contradiction. ~~So~~ f must be continuous.

$\boxed{9}$

(a) Prove that $\sqrt{x}$ is continuous on $\{x \in \mathbb{R} : x \geq 0\}$.

Proof: Let $E = \{x \in \mathbb{R} : x \geq 0\}$ and pick $x_0 \in E$. If f is continuous on E then for every $\varepsilon > 0$ we must find $\delta > 0$ such that if $x \in E$ and $|x - x_0| < \delta$ then $|f(x) - f(x_0)| = |\sqrt{x} - \sqrt{x_0}| < \varepsilon$. Using some algebraic manipulation we see that

$$|\sqrt{x} - \sqrt{x_0}| = \left| \frac{(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})}{(\sqrt{x} + \sqrt{x_0})} \right| = \left| \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \right| \leq \frac{|x - x_0|}{\sqrt{x_0}}$$

If we let $\frac{|x - x_0|}{\sqrt{x_0}} < \varepsilon$ then $|\sqrt{x} - \sqrt{x_0}| < \varepsilon$ for all $x \in \mathbb{R}$ such that $|x - x_0| < \varepsilon \cdot \sqrt{x_0} = \delta$. Thus $\sqrt{x}$ is continuous on E.

(b) Evaluate $\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1}$. Let $f : \mathcal{C}\{1\} \to \mathbb{R}$ where $f(x) = \frac{x-1}{\sqrt{x}-1}$, Clearly, 1 is a cluster point of $\mathbb{R}$.

Using some algebra: $\frac{x-1}{\sqrt{x}-1} \cdot \frac{(\sqrt{x}+1)}{(\sqrt{x}+1)} = \frac{(x-1)(\sqrt{x}+1)}{(x-1)} = \sqrt{x}+1$

Thus, $\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} = \lim_{x \to 1} (\sqrt{x}+1) = \lim_{x \to 1} \sqrt{x} + \lim_{x \to 1} 1 = \sqrt{1} + 1 = 2$

since $\sqrt{x}$, 1 are continuous functions on $\mathcal{C}(\{1\}) \subset \mathbb{R}$, so this follows from the corollary on page 76, and the first paragraph on page 74.

$\boxed{9(c)}$ By exercise 8 we define $\lim\limits_{x \to +\infty} \frac{x}{2x^2+1} = \lim\limits_{y \to 0} g(y)$ where $g:(0,1) \to \mathbb{R}$ and $g(y) = f\left(\frac{1}{y}\right) = \frac{(1/y)}{2(1/y^2)+1} = \frac{y}{2+y^2}$.

Clearly, $\lim\limits_{y \to 0} \frac{y}{2+y^2} = 0$. We now prove this:

For every $\varepsilon > 0$ we must find $\delta > 0$ such that if $y \in (0,1)$ and $|y| < \delta$ then $\left|\frac{y}{2+y^2}\right| < \varepsilon$. But,

$$\Rightarrow \left|\frac{y}{2+y^2}\right| = \frac{|y|}{|2+y^2|} = \frac{|y|}{2+y^2} \leq |y| < \delta.$$ Therefore, if we let $\varepsilon = \delta$ and $|y| < \varepsilon$ then $\left|\frac{y}{2+y^2}\right| < \varepsilon$ as desired.

$\boxed{10}$ (a) $f: E^2 \to \mathbb{R}$ where $f(x,y) = \begin{cases} \frac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

We only need to check the origin for continuity. Consider the convergent sequence: $\lim\limits_{n \to \infty}\left(\frac{1}{n}, 0\right) = (0,0)$

If f is continuous at $(0,0)$ then we must have $\lim\limits_{n \to \infty} f\left(\frac{1}{n}, 0\right) = f(0,0) = 0$ but $f\left(\frac{1}{n}, 0\right) = n^2$ so the limit doesn't exist.

10(b) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ Consider:

$\lim\limits_{n \to \infty}\left(\frac{1}{n}, \frac{1}{n}\right) = (0,0)$ but $\lim\limits_{n \to \infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim\limits_{n \to \infty} \frac{1}{2} = \frac{1}{2} \neq f(0,0)$.

f is not continuous at $(0,0)$

$\boxed{10}$(c) $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0). \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Once again $f(x,y)$ is continuous at all points $(x,y) \neq (0,0)$ so we need to check for continuity at the origin: f is continuous at $(0,0)$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d((x,y),(0,0)) < \delta$ then we have $\left|\dfrac{xy^2}{x^2+y^2}\right| < \varepsilon$. Note that: $(x-y)^2 = x^2 - 2xy + y^2 \geq 0$ which implies that $x^2+y^2 \geq 2xy$. Then we write:

$\Rightarrow \left|\dfrac{xy^2}{x^2+y^2}\right| = \dfrac{|xy^2|}{|x^2+y^2|} \leq \dfrac{|xy^2|}{|2xy|} = \left|\dfrac{xy^2}{2xy}\right| = \left|\dfrac{y}{2}\right| < \varepsilon$ then we have $|y| < 2\varepsilon$ and we may also suppose $|x| < 2\varepsilon$ so that $x^2+y^2 < 8\varepsilon^2 \Leftrightarrow d((x,y),(0,0)) = \sqrt{x^2+y^2} < 2\sqrt{2}\,\varepsilon$.

Therefore, f is continuous at $(x,y) = (0,0)$.

$\boxed{11}$(a) If $f: E \to \mathbb{R}$, $g: E \to \mathbb{R}$ are continuous at $p_0 \in E$, show $(f+g)(p) = f(p) + g(p)$ is continuous at $p_0 \in E$.

$\Rightarrow$ Since f, g are continuous at $p_0 \in E$ we can find $\delta_1, \delta_2 > 0$ such that if $p \in E$ and $d(p, p_0) < \min\{\delta_1, \delta_2\}$ then $|f(p) - f(p_0)| < \frac{\varepsilon}{2}$ and $|g(p) - g(p_0)| < \frac{\varepsilon}{2}$.

Then: $|(f+g)(p) - (f+g)(p_0)| = |(f(p) - f(p_0)) + (g(p) - g(p_0))|$

$\leq |f(p) - f(p_0)| + |g(p) - g(p_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $p \in E$ such that $d(p, p_0) < \min\{\delta_1, \delta_2\}$.

$\boxed{10}$ (b) We now show that if $g(p)$ is continuous at $p_0 \in E$ then $h: E \to \mathbb{R}$, $h(p) = -g(p)$ is continuous at $p_0 \in E$.

$\Rightarrow$ We know for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $p \in E$ and $d(p, p_0) < \delta$ then $|g(p) - g(p_0)| < \varepsilon$.

but, $|g(p) - g(p_0)| = |(-1)\{g(p) - g(p_0)\}| = |(-g(p)) - (-g(p_0))|$

$= |h(p) - h(p_0)| < \varepsilon$. Thus, $h(p)$ is continuous at $p_0 \in E$.

Now to prove $(f-g)(p) = f(p) - g(p)$ is continuous at $p_0 \in E$ simply apply part (a) to $f(p) + (-g(p))$.

$\boxed{10}$ (c) Without loss of generality, we may assume that $f(p_0) \neq 0$, $g(p_0) \neq 0$ since the proof is straightforward if $f(p_0) = g(p_0) = 0$, where $p_0 \in E$.

Now find $\delta_1, \delta_2 > 0$ so that if $|p - p_0| < \delta = \min\{\delta_1, \delta_2\}$ then $|f(p) - f(p_0)| < \dfrac{\varepsilon}{2(|g(p_0)| + \varepsilon)}$ and $|g(p) - g(p_0)| < \min\left\{\varepsilon, \dfrac{\varepsilon}{2|f(p_0)|}\right\}$ where $\varepsilon > 0$ is arbitrary.

Since, $|g(p) - g(p_0)| < \varepsilon$ then $|g(p)| < |g(p_0)| + \varepsilon$.

$$\Rightarrow |(fg)(p) - (fg)(p_0)| = |f(p)g(p) - f(p_0)g(p_0)|$$
$$= |(f(p) - f(p_0))g(p) + (g(p) - g(p_0))f(p_0)|$$
$$\leq |f(p) - f(p_0)||g(p)| + |g(p) - g(p_0)||f(p_0)|$$
$$\leq |f(p) - f(p_0)|(|g(p_0)| + \varepsilon) + |g(p) - g(p_0)||f(p_0)|$$

continued $\rightarrow$

$\boxed{\text{II}}$ (c) continued.

Then: $|(fg)(p)-(fg)(p_0)| < \dfrac{\varepsilon}{2(|g(p_0)|+\varepsilon)}(|g(p_0)|+\varepsilon)+\dfrac{\varepsilon}{2|f(p_0)|}|f(p_0)|$

$$= \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon, \text{ for } d(p,p_0)<\delta.$$

Since $\varepsilon>0$ was arbitrary we find that $f\cdot g$ is continuous at $p_0 \in E$.

$\boxed{\text{II}}$ (d) To prove that $\frac{f}{g}$ is continuous at $p_0 \in E$, we first prove $\frac{1}{g}$ is continuous at $p_0 \in E$ provided that $g(p_0) \neq 0$.

Since $g(p)$ is continuous at $p_0 \in E$ we can pick

$|g(p)-g(p_0)| < \min\left\{ \dfrac{|g(p_0)|}{2}, \dfrac{|g(p_0)|^2\varepsilon}{2} \right\}$ for all

$p \in E$ with $d(p,p_0)<\delta$ for some $\delta>0$.

$\Rightarrow |g(p)| = |g(p_0)-(g(p_0)-g(p))| \geq |g(p_0)|-|g(p)-g(p_0)|$

$> |g(p_0)| - \dfrac{|g(p_0)|}{2} = \dfrac{|g(p_0)|}{2}$. Then we have:

$$\left|\frac{1}{g(p)}-\frac{1}{g(p_0)}\right| = \frac{|g(p)-g(p_0)|}{|g(p)||g(p_0)|} < \frac{|g(p_0)|^2\frac{\varepsilon}{2}}{|g(p_0)|\cdot\left(\frac{|g(p_0)|}{2}\right)} = \varepsilon.$$

since $|g(p)| > \dfrac{|g(p_0)|}{2} \iff \dfrac{1}{|g(p)|} < \dfrac{1}{\frac{|g(p_0)|}{2}}$. Then $\frac{1}{g}$ is continuous at $p_0 \in E$. Now we apply part (c) to $\frac{f}{g} = f\cdot\left(\frac{1}{g}\right)$ to show that $\frac{f}{g}$ is continuous at $p_0 \in E$.

[13] let $f: E \to \mathbb{R}$ be a continuous function on a compact metric space E.

Claim: f is bounded and attains a maximum.

Proof: Assume f is not bounded. Then for each $n = 1, 2, 3, \ldots$ we can find $|f(p_n)| > n$. Since E is compact there exists a convergent subsequence $\{p_{n_k}\}$ of $\{p_n\}$ with

$$\Rightarrow \quad \lim_{k \to \infty} p_{n_k} = p_0, \quad p_0 \in E.$$

Now using the fact that f is continuous implies:

$$\Rightarrow \lim_{k \to \infty} f(p_{n_k}) = f(p_0).$$

Thus we can find a positive integer N such that:

$$\begin{aligned} 1 &> |f(p_{n_k}) - f(p_0)| \\ &\geq |f(p_{n_k})| - |f(p_0)| \\ &> n_k - |f(p_0)| \quad \Rightarrow \quad 1 + |f(p_0)| > n_k \end{aligned}$$

for an ∞ number of positive integers, a contradiction.

Therefore, f is bounded and nonempty, so we can find a sequence in $f(E)$ with:

$$\lim_{n \to \infty} f(q_n) = \text{l.u.b.}\{f(p) : p \in E\}.$$

Once again, since E is compact there exists a convergent subsequence $\{q_{n_k}\}$ of $\{q_n\}$ with:

$$\Rightarrow \lim_{k \to \infty} q_{n_k} = q_0, \quad q_0 \in E.$$

(continued $\Rightarrow$)

13 continued...

but since $\lim_{n\to\infty} f(q_n) = \lim_{k\to\infty} f(q_{n_k})$ we must have $f(q_0) = \text{lub.}\{f(p) : p \in E\}$. Therefore, $f(q_0) \geq f(p)$ for all $p \in E$. Thus, $f(q_0)$ is the maximum value.

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14 (a) Let S be a nonempty compact subset of a metric space E and $p_0 \in E$.

Claim: $\min\{d(p_0, p) : p \in S\}$ exists.

Consider the function $f: E \to \mathbb{R}$ where $f(p) = d(p, p_0)$. By example 2 on page 69, f is continuous on E but by example 7, page 70 f is continuous on the metric space S. Since f is continuous function (real-valued) on the nonempty compact metric space S by corollary 2 on page 78 f attains a minimum at some point $p \in S$ so $\min\{d(p_0, p) : p \in S\}$ exists.

(b) Let S be a nonempty closed subset of E^n and $p_0 \in E^n$.

Claim: $\min\{d(p_0, p) : p \in S\}$ exists.

Note if $p_0 \in S$ then $\min\{d(p_0, p) : p \in S\} = 0$ so we may assume that $p_0 \in \mathcal{C}(S)$. (continued. ⇒)

$\boxed{14}$(b) (continued...)

Now pick $\varepsilon > 0$ such that the closed ball $\overline{B(p_0, \varepsilon)} \subset E^n$ contains points in S, i.e. $\overline{B(p_0, \varepsilon)} \cap S \neq \emptyset$.

$\Rightarrow$ Once again consider the continuous function $f: E^n \to \mathbb{R}$ given by $f(p) = d(p_0, p)$. (Example 2, pg. 69).

Then f is continuous on $\overline{B(p_0, \varepsilon)} \cap S$ as well. (Example 7, pg. 70).

Note that $\overline{B(p_0, \varepsilon)} \cap S$ is a closed and bounded subset of E^n, hence compact.

Then, by corollary 2 on pg. 78 f attains a minimum at some point in $\overline{B(p_0, \varepsilon)} \cap S$. i.e. $\min\{d(p_0, p) : p \in S\}$ exists.

$\boxed{15}$ Let E be a nonempty compact metric space.

<u>Claim</u>: $\max\{d(p, q) : p, q \in E\}$ exists.

<u>Proof</u>: Clearly E is bounded since it is compact and $\{d(p, q) : p, q \in E\}$ is bounded and nonempty.

Then we can find a sequence of points $\{(p_n, q_n)\}_{n=1,2,\dots}$ of E such that:

$$\Rightarrow \lim_{n \to \infty} d(p_n, q_n) = \text{l.u.b.}\{d(p, q) : p, q \in E\}.$$

(continued $\Rightarrow$)

$\boxed{15}$ continued...

Since E is compact there exists convergent subsequences of $\{p_n\}, \{q_n\}$ where $\{p_{n_k}\}, \{q_{n_k}\}$ converge to some $p_0, q_0 \in E$, respectively.

Hence:
$$d(p_0, q_0) = \lim_{k \to \infty} d(p_{n_k}, q_{n_k})$$
$$= \lim_{n \to \infty} d(p_n, q_n)$$
$$= \text{l.u.b.} \{ d(p,q) : p, q \in E \}.$$

Thus, $\max\{d(p,q) : p, q \in E\}$ exists, as desired. $\blacksquare$