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Homework N°7:

Math 4317:

Problem 3:

f_1 restriction of f on S_1 and f_2 restriction of f on S_2. S closed subset of E', therefore:

$$f^{-1}(S) = (f^{-1}(S) \cap S_1) \cup (f^{-1}(S) \cap S_2)$$

$$= f_1^{-1}(S) \cup f_2^{-1}(S) \quad \text{Union of two closed sets}$$

Therefore it is closed and f is continuous.

Problem 6:

boundary $(S) = \overline{S} \cap \overline{eS}$.

we have:

$$\begin{cases} x \in S & : f(x) = 1 \\ x \in eS & : f(x) = 0 \end{cases}$$

Since $S \subset \overline{S}$ and $eS \subset \overline{eS}$, then we have.

$$\begin{cases} x \in \overline{S} & : f(x) = 1 \\ x \in \overline{eS} & : f(x) = 0 \end{cases}$$

$x \in$ boundary $(S) \Rightarrow x \in \overline{S} \cap \overline{eS}$

$\Rightarrow x \in \overline{S}$ and $\nexists x \in \overline{eS}$

$\Rightarrow f$ discontinuous at $x \in$ boun(S).

Please see my remark at the end.

Problem 16:

$$\begin{cases} f: E \to E' \\ f \text{ one-to one and onto.} \\ f \text{ continuous} \\ E \text{ compact.} \end{cases}$$

We want to prove that $f^{-1}: E' \to E$ is continuous.
For this, we show that:
If $S \in E$, S closed, then $f(S) \subseteq E'$ is also closed.

- We have $S \in E$. S is closed if
$p_n \in S$ and $p_n \to p_0$ in E then $p_0 \in S$.

Now let's take a sequence q_n in $f(S)$.
$q_n \in f(S)$, $q_n \to q_0$ in E' and show that $q_0 \in f(S)$.
$q_n = f(p_n)$ with $p_n \in E$.
E is compact, therefore there exists a subsequence
$p_{n_k} \to p_0$.
We have $q_{n_k} = f(p_{n_k})$ therefore $f(p_0) = q_0$
and $q_0 \in f(S)$. so $f(S)$ closed and f^{-1} continuous.

Problem 17:

a) $f(x) = x^2$ not uniformely continuous.
let $\varepsilon > 0$
f uniformely continuous $\Rightarrow \exists \delta$ such that for all
x_1, x_2 with $|x_1 - x_2| < \delta$, $|f(x_1) - f(x_2)| < \varepsilon$

$f(x_1) - f(x_2) = f(x+\delta) - f(x)$

$= \delta^2 + 2\delta x$

$= \delta(2x+\delta)$

for any $\delta > 0$ and $x > \frac{1}{2}\left(\frac{\varepsilon}{\delta} - \delta\right)$

$|f(x_1) - f(x_2)| < \varepsilon$ therefore f not uniformly continuous.

b) $f(x) = \sqrt{|x|}$.

Let's take $x_1, x_2 > 0$.

pick $\varepsilon > 0$.

$|x_1 - x_2| < \delta \Rightarrow |\sqrt{x_1} - \sqrt{x_2}| < \varepsilon$.

We have: $\sqrt{x_1} - \sqrt{x_2} = \dfrac{(\sqrt{x_1} - \sqrt{x_2})(\sqrt{x_1} + \sqrt{x_2})}{(\sqrt{x_1} + \sqrt{x_2})}$

So: $|\sqrt{x_1} - \sqrt{x_2}| = \dfrac{|x_1 - x_2|}{\sqrt{x_1} + \sqrt{x_2}}$

$= |x_1 - x_2|^{1/2} \dfrac{|x_1 - x_2|^{1/2}}{\sqrt{x_1} + \sqrt{x_2}}$

$\dfrac{|x_1 - x_2|^{1/2}}{\sqrt{x_1} + \sqrt{x_2}} < 1 \quad \forall x_1, x_2$, therefore

$|\sqrt{x_1} - \sqrt{x_2}| < |x_1 - x_2|^{1/2}$

So if we pick $\delta = \varepsilon^2$, we have:

$|x_1 - x_2| < \delta \Rightarrow |\sqrt{x_1} - \sqrt{x_2}| < \varepsilon$

and f is uniformly continuous

Problem 19:

We define

$f : E \to \mathbb{R}$ by

$f(p) = d(p, p_0)$.

then $|f(p) - f(p')| = |d(p,p_0) - d(p',p_0)|$

$$\leq d(p,p').$$

therefore f uniformly continuous.

Problem 18:

$f(x) = x$.

let $\varepsilon > 0$.

$|x-y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \delta$

therefore $\forall \varepsilon > 0$, it exists $\delta > 0$ (pick $\delta \leq \varepsilon$)
for which $|x-y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \varepsilon$.

so f uniformely continuous.

Problem ~~19~~ 20:

We have:

$\forall \varepsilon > 0, \exists \delta > 0$ such that:

$|f(x) - f(y)| \leq \delta \Rightarrow |g(f(x)) - g(f(y))| \leq \varepsilon$ ... (1)

and we have:

$\forall \alpha > 0, \exists \beta > 0$ such that:

$|x-y| \leq \beta \Rightarrow |f(x) - f(y)| \leq \alpha$ ... (2)

$\forall \varepsilon > 0$, it will exist some $\delta > 0$ for which (1) is verified and for all values of δ, it will exist some α for which (2) is verified, therefore, $\forall \varepsilon > 0$, it exist $\alpha > 0$ for which: $|x-y| \leq \alpha \Rightarrow |g(f(x)) - g(f(y))| \leq \varepsilon$

so $g \circ f$ is uniformely continuous.

Problem 21:

f uniformly continuous. S dense in E.
$f: S \to E'$ then $\exists g: E \to E'$ also uniformely continuous.

let $\varepsilon > 0$, we know that $\exists \delta > 0$ such that if
$d_S(x_1, x_2) < \delta$ then $d_E(f(x_1), f(x_2)) < \varepsilon$

* for $x \in S$, we define $g(x) = f(x)$.
* for $x \notin S$, we can let a sequence $x_n \to x$ (since S is dense).

Therefore $\{x_n\}$ forms a Cauchy sequence and $\{f(x_n)\}$ is also a Cauchy sequence in E'. Since E' is complete, we have $\{f(x_n)\}$ converges, let's say to y, so we define $g(x) = y$.

We prove that g is uniformely continuous.
$x \in E$, let $\{y_n\}$ a sequence such that $y_n \to x$, we have to show $g(y_n) \to g(x)$.

We have: $|x_n - y_n| = |x_n - x + x - y_n|$
$$\le \underbrace{|x_n - x| + |y_n - x|}_{\le \delta}.$$
both $\{x_n\}$ & $\{y_n\}$ converges to x.

Since f uniformely continuous, then once
$|x_n - y_n| < \delta$ we have $|f(x_n) - f(y_n)| < \varepsilon$
and by our definition of g, we similarly get g uniformly continuous.

Problem 7:

① We suppose $\lim_{x\to a+} f$ and $\lim_{x\to a-} f$ exist and are equal.

$\lim_{x\to a+} f(x) = \lim_{x\to a} f_+(x)$ since f_+ is restriction of f to $U \cap \{x \in \mathbb{R} : x \geqslant a\}$

and $a \in U$, then

$a \in U \cap \{x \in \mathbb{R} : x \geqslant a\}$.

therefore $\lim_{x\to a+} f(x) = \lim_{x\to a} f_+(x) = f_+(a)$.

for the same reason: $\lim_{x\to a-} f(x) = \lim_{x\to a} f_-(x) = f_-(a)$.

Both exist and are equal, ~~a~~ and since f is not defined at a, then they are equal to the limit of f when x goes to a, therefore $\lim_{x\to a} f$ exists.

② We suppose $\lim_{x\to a} f(x)$ exists

let's call it "b" : $\lim_{x\to a} f(x) = b$.

therefore, for an $\varepsilon > 0$ (ε small).

$\lim_{x\to a+\varepsilon} f(x) = b$ and $\lim_{x\to a-\varepsilon} f(x) = b$.

$\Downarrow$ $\Downarrow$

$= \lim_{x\to a+} f(x) = b$ $\lim_{x\to a-} f(x) = b$

$\Rightarrow \lim_{x\to a+} f(x) = \lim_{x\to a^-} f(x)$.

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Problem 12:

We have from p 48 and prob 21 p 63:
If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3 \ldots$ convergent sequences of complex numbers, with limits a and b respectively, then:

$$\lim_{n \to \infty} (a_n + b_n) = a + b \quad \text{---- I}$$

$$\lim_{n \to \infty} (a_n - b_n) = a - b \quad \text{II}$$

$$\lim_{n \to \infty} (a_n b_n) = ab \quad \text{III}$$

and for b and each b_n ~~different~~) nonzero

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{a}{b}. \quad \text{IV}$$

Therefore if we take f and g complex valued functions on E both convergent at a point $p_0 \in E$ & $p_1, p_2, p_3 \cdots$ sequence of complex points of E convergent to p_0, then:

1. from I: ~~limit of~~ the sequence $f(p_1)+g(p_1)$, $f(p_2)+g(p_2), \ldots$ will converge to $f(p_0)+g(p_0)$ and therefore $(f+g)(p_1), (f+g)(p_2) \ldots$ will converge to $(f+g)(p_0)$, so $(f+g)$ continuous at p_0.

Similarly, from II we get $(f-g)$ continuous at p_0, ~~and~~ (fg) continuous at p_0 (from III), from IV (f/g) continuous at p_0 for $g(p_0) \neq 0$.

Remark about problem 6

Let $p \in \partial S$. Claim f is not continuous.

$\partial S = \overline{S} \cap \overline{S^c}$. Since $p \in \partial S$ every

open ball $B(p, \varepsilon)$ contains points of S as

well as S^c. To see this note that

if $B(p, \varepsilon)$ does not contain any points

of S then $B(p, \varepsilon) \subset S^c$. But then

$p \notin \overline{S}$. The same argument shows that

$B(p, \varepsilon)$ must contain points of S^c.

~~Pick pe~~ For every n pick

$p_n \in S \cap B(p, \frac{1}{n})$ and $q_n \in S^c \cap B(p, \frac{1}{n})$.

and consider the sequence

$x_1 = p_1,\ x_2 = q_1,\ x_3 = p_2,\ x_4 = q_2$ etc.

x_n converges to p. But $f(p_n) = 1$

and $f(q_n) = 0$ and ~~the~~ hence

f is not continuous at p.

Conversely if f is not continuous at p there must exist a sequence p_n such that $f(p_n)$ does not converge,

i.e. ~~along~~ there ex a subsequence p_{n_1} with $f(p_{n_1}) = 1$ and a subsequence p_{n_2} $f(p_{n_2}) = 0$, i.e $p_{n_1} \in S$ and $p_{n_2} \in S^c$

Since $p_{n_1} \longrightarrow p \Rightarrow p \in \overline{S}$ and since $p_{n_2} \longrightarrow p \Rightarrow p \in \overline{S^c}$. Thus $p \in \partial S$. $\blacksquare$