HW 8
Solutions

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25) Given a uniformly continuous function, we can, given any $\varepsilon > 0$, find a $\delta > 0$ such that $\forall\, p, q \in E$, $d(p,q) < \delta \Rightarrow d(f(p), f(q)) < \varepsilon$. Hence if we divide the interval $[a,b]$ into intervals of length δ, then by the continuity of the function there is some interval for which $d(p,c) < \delta \Rightarrow d'(f(p), \gamma) < \varepsilon$, where p is a division point. Then choosing $\varepsilon/2$ we can find another point p such that $d'(f(p), \gamma) < \varepsilon/2$. Continuing for ε/n, we can construct a sequence of points whose function values converge to γ. By the compactness of the real interval $[a,b]$, we can choose a convergent subsequence of points with function values converging to γ. Hence for the limiting value $q \in [a,b]$ we have $f(q) = \gamma$.

27) $f: \mathbb{R} \to \mathbb{R}$, $f(x) =$ polynomial of odd degree, with presumably no nonzero coefficients. Any polynomial in $x_1, x_2, x_3, \ldots, x_n$ w/ coefficients in $\mathbb{R}$ is continuous. Since $\mathbb{R}$ is connected, $f(\mathbb{R})$ is connected and we can apply the intermediate value theorem. Any polynomial of odd degree has at least one real root. $\therefore$ There exists $p, q \in \mathbb{R}$ $p < q$ such that $f(p) < 0$ & $f(q) > 0$. Thus we can then choose any $p \in \mathbb{R}$ such that $f(p) \geq 0$ or $f(p) \leq 0$ and obtain any $f(p) \in \mathbb{R}$. Thus $f(\mathbb{R}) = \mathbb{R}$.

30) Consider a circle of points interior to the closed interval in E^2. Assume the function is continuous and 1-1. Then as we traverse the points of the circle either clockwise or counter clockwise, the function must be monotonic increasing or decreasing (otherwise it would not be 1-1), and hence we find for any point on the circle there are nearby points whose function values cannot be made arbitrarily close. This implies discontinuity, a contradiction. Hence the function must not be 1-1.

32) we need $\lim_{n\to\infty} f_n(x) = f(x)$, $x \in \mathbb{R}$. $\{f_n(x)\}_{n=1}^{\infty} = x^{1/2}, (x+x^{1/2})^{1/2}, \ldots$
$\Leftrightarrow f_0(x) = 0,\ f_n(x) = (f_{n-1}(x)+x)^{1/2},\ n = 1,2,3,\ldots$
proof by induction: base case $n=1$. $f_1(x) = (f_0(x)+x)^{1/2} = (0+x)^{1/2} = x^{1/2}$. $f_1(x)$ converges.

Now assume $f_n(x)$ converges at some $n=N$ to $f(x)$. $f_{n+1}(x) = (f_n(x)+x)^{1/2} = (f(x)+x)^{1/2}$. Note that the right hand side is a function of x. Therefore $f_{n+1}(x)$ converges, hence $f_n(x)$ is convergent.

At the limit, the sequence must satisfy $f(x) = (f(x)+x)^{1/2}$, which implies $f^2 = f+x$, and $f^2 - f - x = 0 \Leftrightarrow f(x) = \frac{1 \pm \sqrt{1-4(-x)}}{2}$

$\therefore f(x) = \frac{1 \pm \sqrt{1+4x}}{2}$. Ignoring the negative root, $f(x) = \frac{1+\sqrt{1+4x}}{2}$, $x \geq 0$.

33) b) Conjecturing that the limit is $f(x) = 0$, it needs to be shown that $|0 - x^n(1-x)| < \varepsilon$ for n sufficiently large, independent of x. We need the following fact: $|x^n(1-x)| < \frac{x}{n}$ on $[0,1]$. This can be shown inductively. For $n=1$, $|x - x^2| < |x|$ is a true statement. Then we can assume the truth of the statement for n. Then $x^{n+1}(1+x) = x(x^n)(1-x) < x\left(\frac{x}{n}\right) = x^2/n \leq \frac{x}{n}\ \forall x \in [0,1]$. Now we can say $|0 - x^n(1-x)| = |x^n(1-x)| < \left|\frac{x}{n}\right| < \frac{1}{n}$. This last quantity can be made smaller than ε by choosing $n \geq N = \frac{1}{\varepsilon}$. Then we conclude $x^n(1+x)$ converges uniformly to $f(x) = 0$.

34) Given a sequence of functions f_1, f_2, f_3 on $[0,1]$, determine uniform convergence:

$f_n(x) = \frac{x}{1+nx^2}$; $\lim_{n\to\infty} f_n(x) = 0$

Choose $\varepsilon > 0$, find a N:
$$|f_n(x) - f(x)| = \left|\frac{x}{1+nx^2}\right| \leq \left|\frac{1}{1+n}\right| < \left|\frac{1}{n}\right| < \varepsilon$$
$\Rightarrow N = \frac{1}{\varepsilon}$ $\therefore$ uniformly convergent.

$f_n(x) = \frac{nx}{1+nx^2}$; $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{x^2} = \frac{1}{x}$

Choose $\varepsilon>0$, find a N: $|f_n(x)-f(x)| = \left|\frac{nx}{1+nx^2}-\frac{1}{x}\right| =$
$\left|\frac{nx^2-1-nx^2}{x+nx^3}\right| = \left|\frac{1}{x+nx^3}\right| \leq \left|\frac{1}{1+n}\right| < \left|\frac{1}{n}\right| < \varepsilon$
$\Rightarrow N = 1/\varepsilon$ $\therefore$ uniformly convergent.
$f_n(x) = \frac{nx}{1+n^2x^2}$; $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{x}{2x^2n} = \lim_{n\to\infty}\frac{1}{2nx}=0$
Choose $\varepsilon>0$, find a N: $|f_n(x)-f(x)| = \left|\frac{nx}{1+n^2x^2}-0\right| = \left|\frac{nx}{1+n^2x^2}\right|$
$\leq \left|\frac{n}{1+n^2}\right| < \varepsilon \Rightarrow \frac{N}{1+N^2}=\varepsilon$: $N-\varepsilon N^2-\varepsilon=0 \Rightarrow$
$N = \frac{1\pm\sqrt{1-4\varepsilon^2}}{2\varepsilon}$. N has a very specific condition on ε, $\varepsilon<1/2$.
Since we cannot choose any ε, it is not uniformly convergent.

35) Since f is uniformly continuous, for a given ε, $d'(f(p),f(q))<\varepsilon$ can be achieved for all points $p,q\in E$ by ensuring $d(p,q)<\delta$. Hence, for a given ε, points x and $x+\frac{1}{n}$ are less than the distance δ apart when $|x-x-1/n| = |1/n| < \delta$ or $n>1/\delta$. Note that by the uniform cont. of the f_n, δ does not depend on the point. Then, we can conclude that for $n>1/\delta=N$, $d'(f(x+1/n),f(x))<\varepsilon$. Since the metric of d' is by definition the distance between the maxima of the functions, which will be less than $1/n$ for all $f_n(x)$, $n>N$. Here we supposed $f(x+1/n)$ converged to $f(x)$.

36) $x, x/2, x/3, x/4, \ldots$ attains uniform convergence in $\mathbb{R}$?
$\Rightarrow \{f_n\}_{n=1}^{\infty} = x/n$. $f(x) = \lim_{n\to\infty}\frac{x}{n}=0$.
Choose an $\varepsilon>0$, find an N s.t. $|f_n(x)-f(x)|<\varepsilon$, $\forall n>N$.
$\Rightarrow \left|\frac{x}{n}-0\right| = \left|\frac{x}{n}\right| < \varepsilon \Rightarrow N = \frac{x}{\varepsilon}$. As N must depend on both x and ε, and since this shows f_n converges at different rates depending on x, it is not uniformly convergent.

37) Given the two uniformly convergent sequences of functions f_n and g_n, we know that for all $p \in E$, that when $n > N_1$, $d'(f_n, f) < \varepsilon/2$ and $n > N_2$ ensures $d'(g_n, g) < \varepsilon/2$. Take $N = \max\{N_1, N_2\}$. The assertion is that this N is sufficient to ensure $d'(f_n+g_n, f+g) < \varepsilon$. $d'(f_n+g_n, f+g) = \max\{d(f_n+g_n, f+g) : p \in E\}$ where d is the usual metric of the reals and it is understood that it is relative to our choice of p. Then $\max\{d(f_n+g_n, f+g) : p \in E\}$

$= \max\{|f_n+g_n-(f+g)| : p \in E\} = \max\{|f_n-f+g_n-g| : p \in E\}$

$\leq \max\{|f_n-f| + |g_n-g| : p \in E\} = \max\{d(f_n,f)+d(g_n,g) : p \in E\}$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$ for $n > N$.

The product $f_n g_n$ will also converge uniformly given f_n, g_n, too.
For then $\max\{d(f_n g_n, fg) : p \in E\} = \max\{|f_n g_n - fg| : p \in E\}$

$= \max\{|f_n g_n - f g_n + f g_n - fg| : p \in E\} =$

$= \max\{|(f_n - f) g_n + f(g_n - g)| : p \in E\} < \varepsilon(\max(g) + \varepsilon) +$

$\max(f)\,\varepsilon$, which can be made an arbitrarily small quantity.

38) E, E' metric spaces, $f_n : E \to E'$, f_n bounded, $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent.

<u>To show</u>: The limit function $f(x)$ is bounded.

Note: f_n bounded means that $f_n(E)$ is bounded; i.e. it can be contained in some ball s.t. $|f_n(x)| < M_n$, $M_n \in E'$.

Uniform convergence: for some $\varepsilon > 0$, there is an integer N s.t. $d'(f_n(x), f(x)) < \varepsilon$, $n > N$, all $x \in E$.

Proof: Since $f_n(x)$ is bounded, then for any $x \in E$ and $n \gg N$ with uniform convergence we can state the following:

$$d'(M_n, f(x)) < \varepsilon,$$

although $f_n(x)$ may not attain this bound.
we then take $M = \max\{M_1, M_2, M_3, \ldots\}$.

In order to maintain convergence:

$$d'(M, f(x)) < \varepsilon.$$

However, since M is a constant, this places an upper bound on f(x), the limit function. A similar argument can be made for a lower bound on f(x), hence f(x) is bounded ▨