Remark about problem 27

$f(x)$ an odd degree polynomial.

Can assume that it look like

$f(x) = ax^n + \dots$, $a > 0$. For $a < 0$ the argument is similar.

As $x \to \infty$ $f(x) \to +\infty$ and
as $x \to -\infty$ $f(x) \to -\infty$.

$f(x)$ is continuous. Let $p \in \mathbb{R}$ be any number. There ex. x_1 so that $f(x_1) > p$ and x_2 such that $f(x_2) < p$. By the intermediate value theorem there ex. $c \in \mathbb{R}$ with $f(c) = p$.

Solution of problem 32

$f_0(x) = 0 \quad , \quad f_n(x) = \sqrt{x + f_{n-1}(x)} \quad , \quad x \geq 0$

a) For every $x > 0$, $f_n(x) \leq f_{n+1}(x)$.

Pf: Assume it is true that $f_{n-1}(x) \leq f_n(x)$. Then

$f_n(x) = \sqrt{x + f_{n-1}(x)} \leq \sqrt{x + f_n(x)} = f_{n+1}(x)$

by the monotonicity of the root.

Now $f_0(x) = 0 \leq \sqrt{x} = f_1(x)$.
and by induction the result follows.

b) The sequence $f_n(x)$ is bounded for every $x \geq 0$.

Pretend for the moment that $f_n(x)$ converges for some x. Then by the continuity of the square root and with $f(x) := \lim_{n \to \infty} f_n(x)$,

$f(x) = \sqrt{x + f(x)}$ or

$f^2(x) = x + f(x)$. This leads to

$f(x) = \frac{1}{2} + \sqrt{\frac{1}{4} + x}$ since

only the positive root yields

a non-negative function.

Now we claim that $f_n(x) \leq \frac{1}{2} + \sqrt{\frac{1}{4} + x}$
for all $x \geq 0$ and all $n = 1, 2, 3, \ldots$.

Clearly $f_0(x) = 0 \leq \frac{1}{2} + \sqrt{\frac{1}{4} + x}$.

If $f_n(x) \leq \frac{1}{2} + \sqrt{\frac{1}{4} + x}$, then

$$f_{n+1}(x) = \sqrt{x + f_n(x)} \leq \sqrt{x + \frac{1}{2} + \sqrt{\frac{1}{4} + x}}$$

$$= \frac{1}{2} + \sqrt{\frac{1}{4} + x}$$

and by induction this claim is also
proved to be true.

Thus for every $x \geq 0$, $f_n(x)$ is

a bounded monotone sequence which

therefore converges to some function $f(x) \geq 0$.

We have seen before that

$f(x) = \sqrt{x + f(x)}$ and hence $f(x) = \frac{1}{2} + \sqrt{\frac{1}{4} + x}$