Homework #9.
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Problem 23

Let $\{e_n\}$ be the canonical base of V, and let $x \in V$.

Then $x = \sum_{i=1}^{n} x_i \cdot e_i \Rightarrow \|x\|_1 = \left\|\sum_{i=1}^{n} x_i \cdot e_i\right\|_1 \leq \sum_{i=1}^{n} |x_i| \cdot \|e_i\|_1$ by using the definition of a norm, and $\| \, \|_1$ is any norm.

Applying Cauchy-Schwarz inequality:

$$\|x\|_1 \leq \sum_{i=1}^{n} |x_i| \cdot \|e_i\|_1 \leq \sqrt{\sum_{i=1}^{n} |x_i|^2} \cdot \sqrt{\sum_{i=1}^{n} \|e_i\|_1^2}$$

Notice then that $\sqrt{\sum_{i=1}^{n} |x_i|^2} = \|x\|_e$ and $\sqrt{\sum_{i=1}^{n} \|e_i\|_1^2} = M_1$ where $\| \, \|_e$ is the euclidean norm, and M_1 is some constant.

Then, $\|x\|_1 \leq M_1 \cdot \|x\|_e$

Now we show that $F(x) = \|x\|_1$ is continuous with respect to the euclidean norm.

Let $\varepsilon > 0$, we need to show $\exists \delta_\varepsilon > 0$ s.t $\|x - y\|_e < \delta_\varepsilon \Rightarrow \big\| \|x\|_1 - \|y\|_1 \big\|_e < \varepsilon$.

Let $\delta_\varepsilon = \frac{\varepsilon}{|M_1|}$. Then $\frac{\varepsilon}{|M_1|} \geq \|x - y\|_e \geq \big\| \|x\|_e - \|y\| \big\|_e$

$$\frac{\varepsilon}{|M_1|} \geq \left\| \frac{\|x\|_1}{M_1} - \frac{\|y\|_1}{M_1} \right\|_e = \frac{1}{|M_1|} \cdot \big\| \|x\|_1 - \|y\|_1 \big\|$$

Then, $\varepsilon > \big\| \|x\|_1 - \|y\|_1 \big\|_e \Rightarrow F$ is continuous.

Let $S = \{ x \in V \text{ s.t } \|x\|_e = 1 \}$. S is closed and bounded, so it is compact, and then F has a minimum value in S at x_{min}. Now $\forall x \in V$, $\frac{x}{\|x\|_e} \in S \Rightarrow \left\| \frac{x}{\|x\|_e} \right\|_1 \geq m_1$ where m_1 is the minimum value of F in S. Then, $\|x\|_1 \geq m_1 \cdot \|x\|_e$. Then for any norms, we have shown, that $\exists m_1, M_1, m_2, M_2$ such that

$$m_1 \|x\|_e \leq \|x\|_1 \leq M_1 \cdot \|x\|_e$$
$$m_e \cdot \|x\|_e \leq \|x\|_2 \leq M_2 \|x\|_e$$
$$\Rightarrow \exists\, m = \frac{m_1}{m_2} \text{ and } M = \frac{M_1}{M_2} \text{ s.t } m \leq \frac{\|x\|_1}{\|x\|_2} < M.$$

Then it also follows given that $\mathbb{R}$ is complete, and all norms are equivalent to the Euclidean norm, that the space is complete ($\forall$ norm).

(26)

Given: $a, b \in \mathbb{R}$, $a < b$, f is a continuous real-valued function on $[a, b]$.

Claim: If f is one-one, then $f([a,b])$ is $[f(a), f(b)]$ or $[f(b), f(a)]$, whichever makes sense.

proof of Claim:

To prove this claim, we will first need to prove the following:
subclaim: f is a monotone function.

proof of subclaim:

Since f is continuous and $E = [a,b]$ is compact, $f(E)$ is also compact.
$\Rightarrow \exists$ a maximum and minimum in $f([a,b])$
Since f is one-one, f only attains 1 maximum and 1 minimum.
Since $a < b$, $f(a) < f(b)$ or $f(b) < f(a)$.
$\Rightarrow f$ is strictly increasing or strictly decreasing.

Suppose f is monotone increasing.
Then for $a, b \in \mathbb{R}$, $a < b$, we have
$f(a) < f(b) \Rightarrow f([a,b]) \mapsto [f(a), f(b)]$

Suppose f is monotone decreasing.
Then for $a, b \in \mathbb{R}$, $a < b$, we have
$f(a) > f(b) \Rightarrow f([a,b]) \longmapsto [f(b), f(a)]$. Q.E.D.

Problem 28

An interval of E^n will be of the form: $I_1 \times I_2 \times \ldots \times I_n = I$

where I_i is an interval of $\mathbb{R}$, $i=1,\ldots,n$.

Let I be some interval of E^n (open or closed).

Let $p=(a_1,a_2,\ldots,a_n)$ and $q=(b_1,b_2,\ldots,b_n)$ point of I.

Let m_i be the midpoint of interval I_i, and define:

$$F(T)=\begin{cases}\left((1-2T)\cdot a_1+2Tm_1,\ (1-2T)a_2+2Tm_2,\ \ldots,\ (1-2T)\cdot a_n+2Tm_n\right) & 0\le T<1/2\\ \left((2-2T)\cdot m_1+(2T-1)\cdot b_1,\ (2-2T)m_2+(2T-1)b_2,\ \ldots,\ (2-2T)\cdot m_2+(2T-1)b_n\right) & \frac{1}{2}\le T\le 1.\end{cases}$$

$F:[0,1]\to E^n$.

$F(0)=(a_1,a_2,\ldots,a_n)$ and $F(1)=(b_1,b_2,\ldots b_n)$

$F(1/2)=(m_1,m_2,\ldots,m_n)$ and $\lim\limits_{T\to 1/2} F(T)=(m_1,m_2,\ldots,m_n)$

Hence F is continuous.

If $T<1/2$, $|\Pi_i(F(T))-m_i|=|(1-2T)a_i+2Tm_i-m_i| = |1-2T|\,|a_i-m_i|$.

Since $0<|1-2T|<1$, then $|\Pi_i(F(T))-\ |\le|a_i-m_i| \Rightarrow \Pi_i(F(T))\in I_i \ \forall i=1\ldots n$.

Then, if $0\le T<1/2$, $F(T)\in I$.

If $\frac{1}{2}\le T\le 1$, $|\Pi_i(F(T))-m_i|=|(2-2T)m_i+(2T-1)b_i-m_i|=|1-2T|\ b_i-m_i|$

Then, $F(t)\in I$ if $t\in[0,1]$.

Hence I is arcwise connected $\Rightarrow$ I is connected.

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<u>Definition</u>: A metric space E is arcwise connected if, given any $p, q \in E$, there is a continuous function $f: [0,1] \to E$ s.t. $f(0) = p$, $f(1) = q$.

(a) <u>Claim</u>: An arcwise connected metric space (E) is connected.

<u>proof of claim</u>:

Sps not. That is, sps E is not connected.

Let $E = A \cup B$ s.t. A, B are open subsets of E and $A \cap B = \emptyset$

Since E is arcwise connected, by definition we have $f: [0,1] \to E$ (continuous) where $f(0) = p$, $f(1) = q$.

So, let $p \in A$, $q \in B$.

By proposition on pg 70, since f is continuous and A, B are open, $f^{-1}(A)$ and $f^{-1}(B)$ are open in $[0,1]$.

In fact, $[0,1] = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$

Since $[0,1]$ is connected,

$$f^{-1}(A) \cap f^{-1}(B) = \emptyset \iff f^{-1}(A) = \emptyset \text{ or } f^{-1}(B) = \emptyset$$

WLOG WMA $f^{-1}(A) = \emptyset$.

$\Rightarrow A = \emptyset$.

But $p \in A$. contradiction. Q.E.D.

(b) <u>Claim</u>: Any connected open subset of E^n is arcwise connected.

<u>proof of claim</u>:

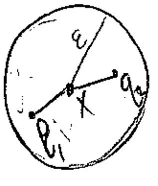
Let S be an open connected subset of E^n.

Then, $S = B(x, \varepsilon) \subset E^n$.

To show S is arcwise connected, we need to show: for $p, q \in S$ $\exists$ continuous function $f: [0,1] \to S$ s.t. $f(0) = p$, $f(1) = q$.

Therefore, we'll need to parametrize any "curve" in S.

Pictoral representation of parametrization:



Suppose $p(t) = (1-2t)q_1 + 2xt \quad 0 \le t \le \frac{1}{2}$

<u>subclaim</u>: $|p(t) - x| \in B(x, \varepsilon)$

<u>proof of subclaim</u>: $|p(t) - x| = |p(t) - (1-2t)x - 2tx|$ $\to$

$$|p(t) - (1-2t)x - 2tx| = |(1-2t)(q - x)|$$
$$= (1-2t)|q_1 - p|$$
$$< (1-2t)\varepsilon < \varepsilon \quad \text{for all } 0 \le t \le \tfrac{1}{2}$$

Now, Suppose $q(t) = (2-2t)x + (2t-1)q_2 \quad \frac{1}{2} \le t \le 1$

subclaim2: $|q(t) - x| \in B(x, \varepsilon)$

proof of subclaim2:

$$|q(t) - x| = |(1-2t)x + (2t-1)q_2|$$
$$= (2t-1)|x - q_2|$$
$$< (2t-1)\varepsilon \le \varepsilon \quad \tfrac{1}{2} \le t \le 1$$

If we let $r(t) = \begin{cases} p(t) & 0 \le t \le 1/2 \\ q(t) & \frac{1}{2} \le t \le 1 \end{cases}$, then $r(t)$ is clearly continuous on $[0,1]$.

$|r(t) - x| < \varepsilon$

$r(0) = q_1 \quad r(1) = q_2 \quad$ Q.E.D.

Problem 41

Let $\varepsilon > 0$. $\{F_n\}$ is point-wise convergent to F.
Let $x \in E$. Then, $\exists N_x \in \mathbb{N}$ s.t $|F_n(x) - F(x)| < \varepsilon/3 \quad \forall n \geq N_x$.

The continuity of F_{N_x} and F is given. Then:
$\exists \delta_x' > 0$ s.t. $d(x, x_0) < \delta_x' \Rightarrow |F_{N_x}(x_0) - F_{N_x}(x)| < \varepsilon/3$
$\exists \delta_x'' > 0$ s.t. $d(x, x_0) < \delta_x'' \Rightarrow |F(x) - F(x_0)| < \varepsilon/3$
Now, choosing $\delta_x = \max\{\delta'_x, \delta''_x\}$.
$|(F_{N_x}(x_0) - F(x_0)) - (F_{N_x}(x) - F(x))| < \frac{2\varepsilon}{3}$
But,
$$|F_{N_x}(x) - F(x)| < \varepsilon/3$$

$\Rightarrow |F_{N_x}(x_0) - F(x_0)| < \varepsilon$

$E = \bigcup_{x \in E} B(x, \delta_x)$, but since E is compact, then $\exists S \subset E$ finite such that $E = \bigcup_{x \in S} B(x, \delta_x)$.

Every x has a corresponding N_x, so let $N = \max_{x \in S}\{N_x\}$

$|F_{N_x}(x_0) - F(x_0)| < \varepsilon \Rightarrow |F_{N_0}(x_0) - F(x_0)| < \varepsilon \quad \forall N_0 \geq N_x$

because $F_N(x_0)$ is always increasing.

Then $\forall x \in E \quad |F_N(x) - F(x)| < \varepsilon$.

Given that $F_n(x)$ is increasing:

$\forall n \in \mathbb{N}$ s.t $n > N \Rightarrow |F_n(x) - F(x)| < \varepsilon \Rightarrow \{F_n\}$ is uniformly convergent.

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Given: E is compact, $p_0 \in E$ is fixed.

$$F : C(E) \longrightarrow \mathbb{R}$$
$$f \longrightarrow f(p_0)$$

Claim: F is uniformly continuous.

proof of Claim:

By definition of uniform continuity, we need to show:
For $\varepsilon > 0$, $\exists$ a real number $\delta > 0$ s.t. for $f, g \in C(E)$
if $d(f,g) < \delta$ we have $d'(F(f), F(g)) < \varepsilon$.

Since f, g are continuous and E is compact,
$f : C(E) \to \mathbb{R}$

$$d'(F(f), F(g)) = |F(f) - F(g)|$$
$$d(f,g) = \max_{x \in E} \{|f(x) - g(x)|\}$$

Let $\delta = \varepsilon$. Then:

$$\max_{x \in E} \{|f(x) - g(x)|\} \geq |f(p_0) - g(p_0)|$$

But $|f(p_0) - g(p_0)| = |F(f) - F(g)|$ and
$\max_{x \in E} \{|f(x) - g(x)|\} < \delta = \varepsilon$.

$\therefore$ $\varepsilon = \delta > \max\{|f(x) - g(x)|\} \geq |f(p_0) - g(p_0)| = |F(f) - F(g)|$

so $|F(f) - F(g)| < \varepsilon$ whenever $\max_{x \in E}\{|f(x) - g(x)|\} < \delta$. Q.E.D.

(44) <u>Given</u>: $\varphi: E \to E'$ is continuous. E, E' are compact metric spaces.

Let Φ denote the map $\Phi: C(E') \to C(E)$
$$f \longrightarrow f \circ \varphi$$

<u>Claim</u>: Φ is uniformly continuous.

<u>proof of claim</u>:

By definition of uniform continuity, we need to show:
For $\varepsilon > 0$ $\exists$ a real number $\delta > 0$ s.t. for $f, g \in C(E')$
$d'(f \circ \varphi(p), g \circ \varphi(p)) < \varepsilon$ whenever $d(f(p), g(p)) < \delta$.

Because f, g are continuous ($f: C(E) \to \mathbb{R}$) and E' compact,

$$d(f(p), g(p)) = \max_{p \in E'} |f(p) - g(p)|$$
$$d'(f \circ \varphi(p), g \circ \varphi(p)) = \max_{p \in E} |f(\varphi(p)) - g(\varphi(p))|$$

Let $\delta = \varepsilon$. Then $\max_{p \in E'} |f(p) - g(p)| < \delta$.

But $\max_{p \in E} |f(\varphi(p)) - g(\varphi(p))| \leq \max_{p \in E'} |f(p) - g(p)| < \delta$

So $\max_{p \in E} |f(\varphi(p)) - g(\varphi(p))| < \varepsilon$. Q.E.D.

Problem 45

$C(E)$ = set of continuous functions $F: E \to \mathbb{R}$.

$\Rightarrow$ $C(E)$ is a well known vector space, and the norm function is given, so $C(E)$ is a normed vector space.

Let $\{F_n\}$ be an arbitrary Cauchy sequence such that $F_i \in C(E)$.

We will prove that $\{F_n\} \to F$ and $F \in C(E)$.

Let $p \in E$. Then $|F_n(p) - F_m(p)| \leq \max_{q \in E} \{ |F_n(q) - F_m(q)| \}$

Then, $|F_n(p) - F_m(p)| \leq \|F_n - F_m\|$.

Since $\{F_n\}$ is a Cauchy sequence:

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t $\|F_n - F_m\| < \varepsilon$ if $n, m > N$.

So, $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t $|F_n(p) - F_m(p)| < \varepsilon$ if $n, m > N$

$\Rightarrow \{F_n(p)\}$ is a Cauchy sequence over the reals.

Given that the reals are complete, $\{F_n(p)\}$ converges to $p' \in \mathbb{R}$.

$\forall p \in E$, define $F: E \to \mathbb{R}$ s.t. $F(p) = p'$.

Now we will prove that $\{F_n\} \to F$.

Let $\varepsilon > 0$.

$\exists N \in \mathbb{N}$ s.t. $\|F_n - F_m\| < \varepsilon/2 \ \forall n, m > N$ since $\{F_n\}$ is a Cauchy sequence.

Now, $|F_n(p) - F(p)| = |F_n(p) - F_m(p) + F_m(p) - F(p)| \leq |F_n(p) - F_m(p)| + |F_m(p) - F(p)|$

Since $F_n(p) \to F(p) \ \forall p \in E$, for each $p \in E$ $\exists M \in \mathbb{N}$ s.t $m > M$

$\Rightarrow |F_m(p) - F(p)| < \varepsilon/2$.

Given also that $|F_n(p) - F_m(p)| = \varepsilon/2$.

Then, $|F_n(p) - F(p)| \leq \varepsilon \Rightarrow \{F_n\} \to F$ since p was arbitrary.

Now we prove that F is continuous.

Let $\varepsilon > 0$.

F_n is continuous, then $\exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |F_n(x) - F_n(x_0)| < \varepsilon/3$.

Also, $F_n(x) \to F(x) \Rightarrow \exists N \in \mathbb{N}$ s.t $|F_n(x) - F(x)| < \varepsilon/3$. $\forall n > \max(N, N')$

$F_n(x_0) \to F(x_0) \Rightarrow \exists N' \in \mathbb{N}$ s.t $|F_n(x_0) - F(x_0)| < \varepsilon/3$

Adding up, we get that $\exists\, \delta > 0$ s.t. if $|x - x_0| < \delta$. Then

$$|F_n(x) - F_n(x_0)| + |F_n(x) - F(x)| + |F_n(x_0) - F(x_0)| < \varepsilon.$$

$$|F(x) - F(x_0)| < \varepsilon \Rightarrow F \text{ is continuous.}$$

In 44 we proved that the map is uniformly continuous. then it is also continuous. So we just need to show that it is a linear operator.

$M: C(E') \longrightarrow C(E)$, and $M(F) = F \circ \varphi$.

$$M(F+g) = (F+g)\circ\varphi = (F+g)(\varphi) = F(\varphi) + g(\varphi) = F\circ\varphi + g\circ\varphi = M(F) + M(g)$$

$$M(cF) = c \cdot F(\varphi) = c \cdot M(F).$$

Problem 46

Let $\{F_n\}$ be a sequence of functions such that $F_i : E \to E'$.

Let $D(F, g) = \text{l.u.b}\{d'(F(p), g(p)) : p \in E\}$

$\{F_n\} \to F \iff \lim_{n \to \infty} D(F_n, F) = 0.$

$\iff \forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t $\forall n > N$ $\text{l.u.b}\{d'(F(p), g(p)) : p \in E\} < \varepsilon.$

$\iff d'(F(p), g(p)) < \varepsilon \quad \forall p \in E.$

Then, $\{F_n\} \to F \iff \{F_n\}$ converges uniformly to F.

Now, E' is complete, so by the proposition on page 86 (IA), $\{F_n\}$ converges uniformly to F. Given that each F_i is continuous, and $\{F_n\}$ uniformly convergent, then F is also continuous.

Also, since each F_i is bounded, by Problem 38 HW#8, F is also bounded, and then the space would be complete.

If F is bounded, but not continuous, then we can use the same result to show that the metric space is also complete.

Moreover, the first space will be contained in the second one.