Final Exam for Anaysis I, Math 4317, December 9, 2008, Allowed time 2 hours and 50 minutes. This is a closed book test. Always state your reasoning otherwise credit will not be given. Try to be as concise and to the point as possible.

Name:

1: (15 points) Give a complete "$\varepsilon - N$" proof for the convergence of the sequence

$$a_n = \frac{n^2+1}{2n^2+n+1} \ , \ n = 1, 2, 3, \dots \ .$$

2: a) Guess the limits of the sequences below and then prove that the sequence indeed converges to that number. State clearly which theorems you use.

$$a) \text{ (10 points) } \ \sqrt{6} \ , \ \sqrt{6+\sqrt{6}} \ , \ \sqrt{6+\sqrt{6+\sqrt{6}}} \ , \ \dots$$

b) (10 points) $\lim_{n\to\infty}(1+\frac{1}{n})^{\sqrt{n}}$

3: Consider a function f from the interval $[0,1]$ to the real numbers.
a) (5 points) What does it mean to say that f is continuous at a point $x_0 \in [0,1]$.

b) (5 points) What does it mean to say that f is continuous on $[0,1]$?

c) (5 points) What does it mean to say that the function is uniformly continuous on the interval $[0,1]$?

d) (5 points) Is there a function that is continuous on $[0,1]$ but not uniformly continuous?

4: Consider the sequence of functions, each defined on the whole real line

$$f_n(x) = \frac{1}{e^{(x-1)n} + 1} \ .$$

a) (10 points) Determine the limit as $n \to \infty$ of this sequence for every x.

b) (5 points) Does it converge uniformly? Justify your answer.

5: (10 points) On $[0, 1]$ consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{q}{q+1} & \text{if } x = p/q,\ p, q \text{ nonnegative integers without common divisor} \end{cases}$$

Show that this function, although it is defined everywhere on $[0, 1]$, does not attain its maximum on $[0, 1]$.

6: a) (10 points) Let $S_1, S_2, S_3, \ldots$ be a sequence of non–empty closed subsets of a compact metric space with the property that $S_{n+1} \subset S_n$ for $n = 1, 2, \ldots$. Show that

$$\cap_{n=1}^{\infty} S_n$$

is not empty.

b) (5 points) Find a counterexample to the above statement if the assumption that the metric space be 'compact' is replaced by the assumption that the metric space is 'closed'

7: (10 points) Show that a compact set S in a metric space E is closed and bounded. You may use the fact that an infinite subset of a compact metric space has at least one cluster point.