Solutions for Practice Test 1 for Anaysis I, Math 4317, September 23, 2010

Always state your reasoning otherwise credit will not be given

1: For any two subsets of S show that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$$

$$x \in A \cap (B \cup C)$$

if and only if x is in A and B or x is in A and C which is the same as saying that $x \in A \cap B$ or $x \in A \cap C$. This is equivalent to $x \in (A \cap B) \cup (A \cap C)$

2 : Let X be a finite set and $f : X \to X$ be a function that is one-one. Show that f is onto.

Consider $f(X) = \{y \in X : y = f(x), x \in X\}$ which is a subset of X, Note that the function f as a function from X to its range $f(X)$ is a one to one function and onto. Hence the elements of X and the elements of $f(X)$ are in one to one correspondence and since $f(X) \subset X$ we must have that $f(X) = X$, i.e, f is onto.

3 : Consider the sequence given recursively by $x_0 = 3$ and

$$x_{n+1} = \frac{1}{3}(x_n + x_n^2)$$

Does this sequence converge?

The sequence x_n is monotone increasing. Clearly, we have that $x_1 = 4 > 3 = x_0$. Further, assuming that $x_n < x_{n+1}$ we find that

$$x_{n+1} = \frac{1}{3}(x_n + x_n^2) < \frac{1}{3}(x_{n+1} + x_{n+1}^2) = x_{n+2} \ .$$

Suppose that the sequence x_n converges to some limit a. Then a must be a solution of $a = \frac{1}{3}(a + a^2)$, or $2a = a^2$, i.e., $a = 0, 2$. On the other hand $a > 3$ which is a contradiction. Hence the sequence does not converge.

4: Consider the sequence

$$a_n = \frac{2n^2 + 3}{n^2 + n + 1} \ .$$

Give a rigorous proof that a_n converges to 2.

$$a_n - 2 = \frac{2n^2 + 3 - 2n^2 - 2n - 2}{n^2 + n + 1} = \frac{1 - 2n}{n^2 + n + 1}$$

Hence for $n \geq 1$

$$|a_n - 2| = \frac{|2n - 1|}{n^2 + n + 1} \leq \frac{2n}{n^2} = \frac{2}{n} \ .$$

Hence for a given $\varepsilon > 0$ we can pick N any positive integer with

$$N > \frac{2}{\varepsilon}$$

which guarantees that $|a_n - 2| < \varepsilon$ for all $n > N$.

5: Prove that if a sequence $a_1, a_2, \cdots$ of real numbers converges to some number a, then the sequence

$$b_n := \frac{\sum_{k=1}^n k a_k}{n^2}$$

also converges. What is the limit of this sequence? Is the converse true?

The converse is not true. Consider the sequence $a_n = (-1)^{n+1}$. Then

$$n > \sum_{k=1}^n k a_k = (1-2) + (3-4) + 5 - \cdots + (-1)^{n+1} n \geq -n$$

Dividing by n^2 and letting n tend to infinity shows that b_n in this example converges to 0, whereas a_n does not converge.

Now assume that a_n converges to a number a. Recall that the sequence a_n must be bounded, i.e., $|a_n| < M$. Pick any $\varepsilon > 0$ then there exists N such that for all $n > N$ $|a - a_n| < \varepsilon/2$. Now consider the sequence

$$c_n := \frac{\sum_{k=1}^n k(a_k - a)}{n^2} = b_n - a\frac{n(n+1)}{n^2}$$

and note that

$$\frac{\sum_{k=1}^n k(a_k - a)}{n^2} = \frac{\sum_{k=1}^N k(a_k - a)}{n^2} + \frac{\sum_{k=N+1}^n k(a_k - a)}{n^2}$$

which implies

$$\begin{aligned} |\frac{\sum_{k=1}^n k(a_k - a)}{n^2}| &< 2M\frac{\sum_{k=1}^N k}{n^2} + \frac{\varepsilon}{2}\frac{\sum_{k=N+1}^n k}{n^2} \\ &= M\frac{N(N+1)}{n^2} + \frac{\varepsilon}{4}\frac{n(n+1) - N(N+1)}{n^2} \\ &\leq M\frac{N(N+1)}{n^2} + \frac{\varepsilon}{4}\frac{n(n+1)}{n^2} \ . \end{aligned}$$

Now pick n larger than N so that inaddition the fist term is less than $\varepsilon/2$. The second term is also less than $\varepsilon/2$ and hence the sequence c_n converges to zero. Thus, the sequence b_n converges to $a/2$.

6: Find a metric space and a sequence of bounded closed sets $S_i, i = 1, 2, \cdots$ such that

$$S_1 \supset S_2 \supset S_3 \cdots$$

but

$$\cap_{i=1}^{\infty} S_i = \emptyset$$

Consider the metric space given by the rational numbers Q with the distance given by $d(p,q) = |p-q|$. Then $S_n = Q \cap [\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n}]$ for each positive integer n is a closed set. Indeed if , for fixed n, $p_k \in S_n$ is a sequence of rationals that converges to some $p \in Q$, then the limit p is also in $[\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n}]$, since $[\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n}]$ is closed in the reals. Thus, S_n is closed for each n. Clearly, $S_{n+1} \subset S_n$ for $n = 1, 2, 3, \ldots$ and S_n is bounded for each n. However,

$$\cap_{n=1}^{\infty} S_n = \emptyset$$

since $\sqrt{2}$ is not a rational number.

7: Prove that every bounded monotone sequence of real numbers converges.

We assume that the sequence is monotone increasing and bounded from above, the other case is analogous. Let $a_1 \le a_2 \le a_3 \cdots$ be the sequence. Then our candidate for the limit is

$$a := l.u.b(\{a_1, a_2, a_3, \cdots\})$$

Pick any $\varepsilon > 0$. Since $a + \varepsilon$ is an upper bound we have

$$a_n < a + \varepsilon$$

for all n. since $a - \varepsilon$ is not an upper bound, there must be a first N with $a_N > a - \varepsilon$. Since the sequence is monotone increasing we have that for all $n > N$

$$a - \varepsilon < a_n < a + \varepsilon \ .$$

8 : Let A and B be subsets of a metric space. Assume that A is closed and B is open. Show that the complement of $A \cap B$ in A is closed.

The complement of $A \cap B$ in A consists of all points in A that are not in B, i.e., it is the set $A \cap B^c$ where B^c is the complement of B in the metric space. Since B^c is closed and A is closed so is $A \cap B^c$.

9 : Find a collection of nonempty closed subsets of the real numbers whose union is bounded and open. For every positive integer n consider the set

$$S_n = [-2 + \frac{1}{n}, 2 - \frac{1}{n}] \ .$$

These sets are closed and nonempty. Every point in $x \in (-2, 2)$ is in $\cup_{n=1,2,3,\ldots} S_n$ but neither -1 nor 2. Hence

$$\cup_{n=1,2,3,\ldots} S_n = (-2, 2) \ .$$

10: Is the set consisting of all rational numbers r with $0 \le r \le 1$ a compact subset of the real numbers?

No. Any compact subset of the reals must be closed, but the rationals in the interval $[0, 1]$ are not closed.