Review problems

Math 4317 / Dec. 6/08

1) Prove that every bounded monotone sequence has a limit. You have to reduce the problem to the l.u.b and g.l.b property of the reals.

2) Related to the above is: Show that every bounded monotone sequence is a Cauchy sequence.

3) Prove that for two bounded set of real numbers A, B with $A \subset B$,

$$\text{l.u.b.}\, A \leq \text{l.u.b.}\, B$$
$$\text{g.l.b.}\, A \geq \text{g.l.b}\, B$$

4) For any bounded sequence of real numbers $a_1, a_2, a_3, \dots$ define

$$b_k = \text{l.u.b.}\{a_n : n \geq k\} = \text{l.u.b.}\{a_n, a_{n+1}, \dots\}$$

and

$$c_k = \text{g.l.b.}\{a_n, a_{n+1}, a_{n+2}, \dots\}.$$

Show that both, the sequence $b_1, b_2, \dots$ and $c_1, c_2, \dots$ converge.

$$\lim_{k\to\infty} b_k =: \limsup_{k\to\infty} a_k, \quad \lim_{k\to\infty} c_k =: \liminf_{k\to\infty} a_k.$$

5) Use 4) to show that if

$$\limsup_{k\to\infty} a_k = \liminf_{k\to\infty} a_k = a$$

then the sequence a_k converges to a.

6) Use 5) to show that every Cauchy sequence of real numbers converges.

7) Let $f(x)$ be a non-negative monotone decreasing function. Show that

$$a_n = \sum_{k=1}^{n} f(k) - \int_1^n f(x)dx$$

converges.

8) Show that for $a \in \mathbb{R}$, $|a|<1$, for any $k \geq 0$, integer,

$$\lim_{n\to\infty} n^k a^n = 0 .$$

9) This one is tricky! Let $\{a_m\}_{m=1}^{\infty}$ be a sequence of real numbers, so that $\frac{a_m}{m}$ is bounded below and with

$$a_{m+n} \leq a_m + a_n \quad \text{all } m,n \in \mathbb{N}.$$

Show that $\frac{a_n}{n}$ converges as $n \to \infty$.

10) Evaluate $\lim\limits_{n\to\infty} \left(1+\frac{1}{n^{1+\alpha}}\right)^n, \ \alpha > 0$

$\lim\limits_{n\to\infty} \left(1+\frac{1}{n^{1-\alpha}}\right)^n, \ \alpha \geqslant 0.$

Integrals

11) Is

$$f(x) = \begin{cases} 0 & x \text{ irrational} \\ \frac{1}{q} & x = \frac{p}{q}, \ p, q \text{ no common divisors} \end{cases}$$

integrable on $[0,1]$ and if yes: what is

$$\int_0^1 f(x)\,dx = ?$$

12) For which values of α does the improper integral

$$\int_2^\infty \frac{1}{x} \frac{1}{(\log x)^\alpha} dx \quad \text{exist?}$$

Discuss also

$$\int_0^{\frac{1}{2}} \frac{1}{x} \frac{1}{(-\log x)^\alpha} dx$$

13) Let $f(x)$ be a non-negative function, decreasing, so that

$\lim\limits_{L\to\infty} \int_0^L f(x)\,dx$ exists. Show that $\lim\limits_{x\to\infty} x f(x) = 0$

14) Find a function $f(x)$, $f(x) \geq 0$ such that

$$\lim_{L \to \infty} \int_0^L f(x)\,dx$$ exists. but so that

$$\lim_{x \to \infty} f(x)$$ does not exist.

15) Compute

$$\lim_{N \to \infty} \sum_{k=1}^{N} \frac{1}{Nx + ky}, \quad x, y \text{ real}, \; x, y > 0.$$

Functions

16) $$f(x) = \begin{cases} 2x-1 & x \text{ rational} \\ 5-x & x \text{ irrational} \end{cases}$$

Discuss the continuity of f.

17) Is $f(x)$ above integrable on $[0,1]$?

18) Consider the functions

$$f_n(x) = (1-x)x^n, \quad g_n(x) = n(1-x)x^n, \quad h_n(x) = n^2(1-x)x^n$$

on $[0,1]$.

a) Determine the limits as $n \to \infty$.

b) Which of the sequences is uniformly bounded?

c) Which of the sequences converges uniformly?

d) Sketch roughly the behavior of these sequences.

19) Let E be a metric space and $f_n : E \to \mathbb{R}$ a sequence of continuous functions that converges uniformly to some $f: E \to \mathbb{R}$. Prove that f is continuous.

20) Assume that $f: [0,1] \to \mathbb{R}$ is bounded and monotone. Show that f has at most a countable number of discontinuities.

21) Show: $f: E \to E'$ is uniformly continuous if and only if for every sequence $x_n, y_n \in E$ with $d(x_n, y_n) \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} d'(f(x_n), f(y_n)) = 0.$$

Open, closed and compact sets

22) Prove that $[0,1]$ is compact in the sense that every sequence in $[0,1]$ has a convergent subsequence. (Infinite pidgeon-hole principle).

23) Is a subset of a compact set compact? Discuss $\mathbb{Q} \cap [0,1]$.

24) Is a closed subset of a compact set compact?

25) A sequence $p_n \in E$, which has finitely many cluster points forms a ~~is~~ compact set.

26) E metric space, $S \subset E$ a subset. The closure $\overline{S}$ of S is defined to be

$$\overline{S} = \bigcap_{\substack{F \supset S \\ F \text{ closed}}} F$$

Show that $\overline{S}$ is the set of all limits of sequences in S that converge in E.

27) $F \subset E$ closed. Show that if $p_n \in F$ and $\lim_{n \to \infty} p_n = p$ in E, then $p \in F$.

Conversely, if for every sequence $p_n \in F$

$\lim_{n\to\infty} p_n = p \in F$, then show that F is closed.

28) Recall that for $S \subset E$, $p \in S$ is an interior point if there ex. $\varepsilon > 0$ so that the ball $B_\varepsilon(p) \subset S$. The open interior of S is

$$S^\circ = \{ p \in S : p \text{ is an interior point} \}$$

Show that

$$S^\circ = \bigcup_{\substack{O \subset S \\ O \text{ open}}} O .$$

29) $\partial S := \overline{S} \cap \overline{S^c}$

Show that $E = \partial S \cup S^\circ \cup (S^c)^\circ$.

Differentiation of functions

30) State and prove the mean value theorem.

31) Solve problem 13 on p 110 of I.A.

32) Graph the function $x \log x$, $0 < x < \infty$. Show that $x \log x > x - 1$ all $x > 0$.