Test 2 for Anaysis I, Math 4317, October 29, 2008, Allowed time 50 minutes. This is a closed book test.

Always state your reasoning otherwise credit will not be given. Try to be as concise and to the point as possible.

Name:

1: a) (10 points) Let (E, d) and (E', d') be metric spaces and assume that $f : E \to E'$ is a function that satisfies

$$d'(f(p), f(q)) \leq d(p, q)^{1/2}$$

for all $p, q \in E$. Show that f is continuous. Is it uniformly continuous?

b) (10 points) Let E be a metric space with the metric

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q. \end{cases}$$

Show that every real valued function f on (E, d) is continuous.

2: Consider the sequence of functions $f_n : [0, 1] \to \mathbf{R}$ given by $f_n(x) = (1 - x)x^n$.
a) (5 points) Show that this sequence of functions converges and determine the limiting function.

b) (5 points) Is the convergence uniform? What would you guess?

c) (10 points) Prove the guess you have made in b).

3: Consider the sequence of functions $f_n : [0, 1] \to \mathbf{R}$ given by $f_n(x) = n(1 - x)x^n$.
a) (5 points) Show that this sequence of functions converges and determine the limiting function.

b) (5 points) Is the convergence uniform? What would you guess?

c) (10 points) Prove the guess you have made in b).

4: (10 points) Let E be a metric space and let $B \subset E$ be a subset with the property that for every family of closed subsets $F_j, j \in J$ with $\cap_{j \in J} F_j \subset B$, there exists a finite subfamily $I \subset J$ so that $\cap_{j \in I} F_j \subset B$. What can you say about the complement of B in E?

5: (10 points) Consider the sequence of functions on $\{x \in \mathbf{R} : x \geq 0\}$

$$\sqrt{x}\ ,\ \sqrt{x + 2\sqrt{x}}\ ,\ \sqrt{x + 2\sqrt{x + 2\sqrt{x}}}\ ,\ \ldots$$

Prove that this sequence converges for every x and find the limit.

6: Let E, E' be two metric spaces and assume that $f : E \to E'$ is a continuous function. True or False: (5 points each)

a) If $S \subset E$ is compact, then $f(S) \subset E'$ is also compact.

b) If $S \subset E$ is closed then $f(S) \subset E'$ is closed.

c) If $T \subset E'$ is compact then $f^{-1}(T) \subset E$ is compact.

d) If $T \subset E'$ is open then $f^{-1}(T) \subset E$ is open.