Test 1 for Anaysis I, Math 4317, September 29, 2010, Allowed time is 50 minutes. This is a closed book test.

Always state your reasoning otherwise credit will not be given. Try to be as concise and to the point as possible.

Name:

1: (10 points) Consider the subset S of the real numbers given by

$$S = (-1, 1) \cup \{2\} .$$

a) Find the closure $\overline{S}$, i.e., the smallest closed set that contains S.

$$\overline{S} = [-1, 1] \cup \{2\} .$$

b) Find the open interior of S, S°, i.e., the largest open subset of S.

$$S^\circ = (-1, 1) .$$

2 :(15 points) For which real numbers x is the following inequality true:
a) $x^2 + 2x - 3 < 0$

$$x^2 + 2x - 3 = (x + 3)(x - 1) < 0$$

holds for $-3 < x < 1$.
b) Show that the x axis is a closed subset of the plane.

First solution: The complement of the x axis is given by $\{(x, y) \in E^2 : y \neq 0\}$. For every point $p = (x, y)$ in the complement we know that the open ball $B(p, |y|)$ is also in the complement. Hence the complement is open and the x-axis is closed.

Second solution: Let $p_n = (x_n, 0)$ be a sequence of points on the x axis which converges in E^2 to some point $p_0 = (x, y)$. Since the y component of the sequence p_n is the sequence consisting of zeros, the limit is zero and hence $y = 0$ and $p_0 = (x, 0)$ which is on the x axis. Hence the x axis is closed.

3 :(15 points) Consider the sequence given recursively by $x_0 = 1$ and

$$x_{n+1} = \frac{1}{3}(x_n + x_n^2) \ , \ n = 1, 2, \cdots$$

Is the sequence monotone? Is it bounded? Does this sequence converge? If it does, what is its limit?

$$x_1 = \frac{2}{3}$$

Via induction we show that the sequence x_n is decreasing. Assuming that $x_n < x_{n-1}$ we have

$$x_{n+1} = \frac{1}{3}(x_n + x_n^2) < \frac{1}{3}(x_{n-1} + x_{n-1}^2) = x_n \ .$$

Since the sequence is decreasing and positive, it converges. The limit x has to satisfy

$$x = \frac{1}{3}(x + x^2)$$

or

$$2x = x^2 \ .$$

$x = 2$ is not a possible limit and hence $x = 0$.

4: (15 points) Consider the subset S of the real numbers consisting of all series of the form

$$\sum_{k=1}^{\infty} \frac{a_k}{2^k}$$

where $a_k \in \{0, 1\}$. What is the least upper bound of the set S?

Every sequence satisfies

$$\sum_{k=1}^{\infty} \frac{a_k}{2^k} \le \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 \ .$$

Since the sequence

$$\sum_{k=1}^{\infty} \frac{1}{2^k}$$

is in S, 1 is l.u.b(S).

5: (20 points) Let E be a compact metric space. Prove that if $S_i, i \in I$ is a collection of closed subsets of E such that for any finite sub-collection $J \subset I$

$$\cap_{i \in J} S_i \neq \emptyset \ ,$$

then

$$\cap_{i \in I} S_i \neq \emptyset \ .$$

Hint: Reformulate the problem by taking complements in E.

By taking complements the problem is equivalent to showing the following: If for any $J \subset I$, finite

$$\cup_{i \in J} S_i^c \neq E$$

then

$$\cup_{i \in I} S_i^c \neq E \ .$$

Suppose that

$$\cup_{i \in I} S_i^c = E \ ,$$

then, since for every $i \in I$ S_i is closed, $\{S_i^c\}_{i \in I}$ is an open cover. Hence there exists a finite sub-cover $J' \subset I$ with

$$\cup_{i \in J'} S_i^c = E \ ,$$

which is a contradiction.

6: Let E be a metric space. True or false: (each item counts 5 points)

a) The intersection of an arbitrary collection of closed subsets of E is closed. **TRUE**

b) Any closed subset of a compact subset of E is compact. **TRUE**

c) Bounded and closed subsets of E are compact. **FALSE**

d) Every compact metric space is complete. **TRUE**

e) A subset of a complete metric space is complete. **FALSE**