A good source for this material is the book by Reed and Simon, Methods of Modern Mathematical Physics, Vol. I on Functional Analysis, which we follow.

1 The Diagonal Argument

1.1 DEFINITION (Subsequence)**.** A subsequence of a given sequence is a function $m : \mathbb{N} \to \mathbb{N}$ which is strictly increasing.

1.2 THEOREM. *Consider a sequence of functions $\{f_n(x)\}_N^\infty$ defined on the positive integers that take values in the reals. Assume that this sequence is uniformly bounded, i.e., there is a positive constant such that*

$$|f_n(x)| \le C$$

for all $n = 1, 2, \dots$ and all $x \in \mathbb{N}$. Then there exists a subsequence $m(j)$ such that $f_{m(j)}$ converges for all $x \in \mathbb{N}$.

Proof. Since $f_n(1)$ is a bounded sequence, there exists a subsequence $f_{n_1(j)}$ of functions such that $f_{n_1(j)}(1)$ converges as $j \to \infty$. Now we pick a subsequence of $n_1(j)$ which we call $n_2(j)$ such that the sequence of functions $f_{n_2(j)}(x)$ converges for $x = 2$. Proceeding in an inductive fashion we obtain a subsequence $n_k(j)$ of the sequence $n_{k-1}(j)$ such that the for the sequence of functions $f_{n_k(j)}(x)$, $f_{n_k(j)}(k)$ is convergent. Note, that this construction guarantees that $f_{n_k(j)}(r)$ converges for all $r \le k$. Now we set

$$m(j) = n_j(j) \ ,$$

i.e., we pick the 'diagonal sequence'. Note that $f_{m(j)}(k)$ converges for every k, since the sequence

$$f_{m(k)}(k) \ , \ f_{m(k+1)}(k) \ , \ f_{m(k+2)}(k) \ , \ f_{m(k+3)}(k) \ \dots$$

is a subsequence of the sequence $f_{n_k(j)}(k)$, which converges. Hence $f_{m(j)}(k)$ converges for all $k = 1, 2, 3, \dots$. For every k, there are finitely many terms that are not part of the subsequence $f_{n_k(j)}(k)$, namely

$$f_{m(1)}(k) \ , \ f_{m(2)}(k) \ , \ f_{m(3)}(k) \dots f_{m(k-1)}(k) \ ,$$

but they are immaterial for the convergence of the sequence. □

2 The $\varepsilon/3$ argument

2.1 THEOREM. *The space $C([0,1])$ consisting of continuous functions $f : [0,1] \to \mathbb{R}$ with metric*

$$D(f,g) = \max_{0 \le x \le 1} |f(x) - g(x)|$$

is a complete metric space.

Proof. We have learned before that $C([0.1])$ is a metric space. We have to worry about completeness. Let $f_n(x) \in C([0,1])$ be a Cauchy Sequence. Thus, for every $\varepsilon > 0$ there exists N such that for all $n, m > N$

$$\max_{0 \le x \le 1} |f_n(x) - f_m(x)| < \varepsilon/2 \ .$$

In particular, for every fixed $x \in [0,1]$, $f_n(x)$ is a Cauchy Sequence of real numbers and since the reals are complete, this sequence has a limit which we denote by $f(x)$. Since for any m

$$|f(x) - f_m(x)| = \lim_{n \to \infty} |f_n(x) - f_m(x)| \le \text{l.u.b}\{|f_n(x) - f_m(x)| : n > N\} \ ,$$

we have that for all $m > N$

$$|f(x) - f_m(x)| \le \varepsilon/2 < \varepsilon \ . \tag{2.1}$$

Note that x is arbitrary and that ε is independent of x, i.e., the convergence is uniform. Although we know from previous arguments that the limit must be continuous, let us prove this, because this uses a typical $\varepsilon/3$ argument. $\varepsilon > 0$. We have seen that there exists N so that for all $n > N$ and all $x \in [0,1]$,

$$|f(x) - f_n(x)| < \varepsilon/3$$

Fix such a value for n and fix x. Since f_n is continuous, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in [0,1]$ with $|x - y| < \delta$ we have that

$$|f_n(x) - f_n(y)| < \varepsilon/3 \ .$$

Since we also have that

$$|f_n(x) - f_n(y)| < \varepsilon/3 \ ,$$

we may use the triangle inequality

$$|f(x) - f(y)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(x) - f_n(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

Thus, for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever y is such that $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Thus, the limit f is continuous. Note, that from (2.1) we know that for any $\varepsilon > 0$ there exists N such that for all $n > N$

$$|f(x) - f_n(x)| \le \varepsilon/2$$

and hence

$$D(f, f_n) = \max_{0 \le x \le 1} |f(x) - f_n(x)| \le \varepsilon/2 < \varepsilon ,$$

and hence the sequence f_n converges to f in the metric $D(f, g)$. □

3 Equicontinuity and the Theorem of Arzela-Ascoli

We have seen various notions of continuity but they all were statements about a single function. In this section we shall talk about the continuity properties of a family of functions. In what follows we shall always consider two metric spaces E, E' and $\mathcal{F}$ a family of continuous functions from E to E'.

3.1 DEFINITION. A family $\mathcal{F}$ of functions from E to E' is **equicontinuous** if for every $\varepsilon > 0$ and for every $p \in E$ there exists $\delta > 0$ such that for all $f \in \mathcal{F}$

$$d'(f(p), f(q)) < \varepsilon$$

whenever $d(p, q) < \delta$.

Note that the point here is that δ depends only on p and ε but not on the function under consideration. Here is a simple result that gives you a bit of a feeling what this notion accomplishes.

3.2 THEOREM. *Let f_n, n=1,2,3 ... be a sequence of functions from E to E' with the property that $f_n(p)$ converges to $f(p)$ for every $p \in E$. Suppose further that the family $\{f_n\}_{n=1}^{\infty}$ is equicontinuous. Then f is continuous, and moreover, the family $\{f, f_1, f_2, \dots\}$ is also equicontinuous.*

Proof. Fix any ε and fix any $p \in E$. Then there exists $\delta > 0$ such that whenever $d(p, q) < \delta$, $d'(f_n(p), f_n(q)) < \varepsilon/3$ for all $n = 1, 2, 3, \dots$. Further there exists N such that both, $d'(f(p), f_n(p)) < \varepsilon/3$ and $d'(f(q), f_n(q)) < \varepsilon/3$ for all $n > N$. Fix such a value for n. Then for all q with $d(p, q) < \delta$ we have that

$$d'(f(p), f(q)) \le d'(f(p), f_n(p)) + d'(f_n(p), f_n(q)) + d'(f_n(q), f(q)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .$$

Note that since we know that whenever $d(p,q) < \delta$, then $d'(f_n(p), f_n(q)) < \varepsilon/3 < \varepsilon$ we know that the family $\{f, f_1, f_2, \dots\}$ is also an equicontinuous family. □

Another simple consequence is the following

3.3 THEOREM. *Let $\{f_n\}_{n=1}^{\infty}$ be an equicontinuous family of functions from E to E'. Assume that E' is complete and that $f_n(p)$ converges for all $p \in D$ where $D \subset E$ is dense. Then $f_n(p)$ converges for all $p \in E$.*

Proof. Recall that $D \subset E$ dense means that for every $p \in E$ and every $\varepsilon > 0$ there exists $q \in D$ such that $d(p,q) < \varepsilon$. Now pick $p \in E$ arbitrary and pick an $\varepsilon > 0$. There exists $\delta > 0$ such that for all $q \in E$ with $d(p,q) < \delta$ we have for all $n = 1, 2, 3, \dots$ $d'(f_n(p), f_n(q)) < \varepsilon/3$. In particular there exists $q \in D$ with $d(p,q) < \delta$. Since $f_n(q)$ converges for $q \in D$ there exists N so that for all $n, m > N$, $d'(f_n(q), f_m(q)) < \varepsilon/3$ and hence

$$d'(f_n(p), f_m(p)) \le d'(f_n(p), f_n(q)) + d'(f_n(q), f_m(q)) + d'(f_m(q), f_m(p)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \, .$$

Thus, $f_n(p)$ is a Cauchy sequence in E' and since E' is complete it converges. Thus $f_n(p)$ converges for all $p \in E$. □

If in the definition of equicontinuity, δ does only depend on ε and not on the point $p \in E$, then we call the family $\mathcal{F}$ uniformly equicontinuous. More precisely we have

3.4 DEFINITION. A family $\mathcal{F}$ of functions from E to E' is **uniformly equicontinuous** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{F}$ and all p, q with $d(p,q) < \delta$ it follows that

$$d'(f(p), f(q)) < \varepsilon \, .$$

Here is a first interesting theorem concerning uniform equicontinuity.

3.5 THEOREM. *Let $\{f_n\}_{n=1}^{\infty}$ be a uniformly equicontinuous family of real valued functions on the interval $[0,1]$. Assume further that $f_n(x)$ converges to $f(x)$ for all $x \in [0,1]$. Then the convergence is uniform.*

Proof. Pick $\varepsilon > 0$. By Theorem 3.2 we kow that the limiting function is continuous and that the family $\{f, f_1, f_2, \dots\}$ is equicontinuous. There exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/3$ and $|f_n(x) - f_n(y)| < \varepsilon/3$ for all n, whenever $|x - y| < \delta$. Now consider the points $x_1, \dots, x_M$ so that no point $x \in [0,1]$ is farther away from x_j for some $j = 1, 2, \dots, M$. This is a finite set of points and hence there exists N, depending only on ε such that for all $n > N$ and all $j = 1, \ldots, M$,

$$|f(x_j) - f_n(x_j)| < \varepsilon/3 \ .$$

For any $x \in [0,1]$ we have therefore for some x_j with $|x - x_j| < \delta$ that

$$|f(x) - f_n(x)| \leq |f(x) - f(x_j)| + |f(x_j) - f_n(x_j)| + |f_n(x_j) - f_n(x)|$$

and since each term is strictly less than $\varepsilon/3$ the result follows. □

We are now ready to formulate and prove a central result.

3.6 THEOREM (Arzela-Ascoli Theorem). *Let $\{f_n\}_{n=1}^{\infty}$ be a uniformly equicontinuous family of uniformly bounded functions on* $[0,1]$. *Then there exists a subsequence $f_{n(i)}$ which converges uniformly on* $[0,1]$.

Proof. The rational numbers in $r_m \in [0,1]$ are countable and dense. Since the functionsf_n are uniformly bounded we also know that $|f_n(r_m)| \leq C$ for some constant $C > 0$. From the 'Diagonal argument' we know that there exists a subsequence $n(i)$ such that $f_{n(i)}(r_m)$ converges for all r_m. By Theorem 3.3 we know that the sequence $f_{n(i)}(x), i = 1, 2, 3 \ldots$ converges for all $x \in [0,1]$ to some function $f(x)$. By Theorem 3.2 we know that this function is continuous and by Theorem 3.5 we know that the convergence is uniform. □